LECTURE 27
MATH 229

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1. Stable case

1.1. **Setup and statement of the theorem.** Today we will (hopefully) finish the stable case.

Let M be an ω-stable (ω-categorical) structure. We didn't define ω-stable, so we will restrict to the case that M is finitely homogeneous and stable, but everything we say will be true in this generality. In particular, the features of ω-stability that we will use are the following:

(i) M is ranked (in fact we will assume finite rank[1]).
(ii) If we have a definable subset $X \subseteq M^k$ which has rank n, there is $X_0 \subseteq X$ of rank n and *indivisible* in the sense that it cannot be split into two disjoint sets of rank n.
(iii) If X is defined over a X_0 cannot be defined over $\operatorname{acl}^{eq}(a)$.
(iv) If $\varphi(\overline{x}, \overline{y})$ is a formula there is a *normalization* $\varphi^*(\overline{x}, \overline{y})$ such that
   - $\operatorname{rank}\left(\varphi(\overline{x}, \overline{y}) \Delta \varphi^*\left(\overline{a}, \overline{b}\right)\right) < n$
   - $\forall \overline{b}, \overline{b'} \models q$

$$\operatorname{rank}\left(\varphi^*\left(\overline{x}, \overline{b}\Delta\right) \varphi^*\left(\overline{x}, \overline{b'}\right)\right) < n \qquad \Longrightarrow \qquad \varphi^*\left(\overline{x}, \overline{b}\right) = \varphi^*\left(\overline{x}, \overline{b'}\right)$$

Let P be a $\emptyset$-definable set M^{eq}. Say that P coordinatizes M if for all $a \in M$

$$\operatorname{acl}(a) \cap P \neq \emptyset \ .$$

Theorem 1. *Let M be ω-stable, ω-categorical, (of finite rank). There exists is a rank 1 set (finite rank, strongly minimal) which coordinatizes M.*

Corollary 1. *If M is primitive then M is a Grassmannian over a (finite union of) strongly minimal set(s).*

1.2. **Proof of the theorem.**

1.2.1. *Aside on strongly minimal.* Let X, Y be $\emptyset$-definable, strictly minimal (i.e. $\operatorname{acl}(a) = \{a\}$) then either

- there is a unique $\emptyset$-definable bijection between X and Y, or
- X, Y are orthogonal: $\overline{a} \in X^k$, $\overline{b} \in Y^l$, $\overline{a} \mathrel{\unicode{x2AEB}} \overline{b}$.

Proof. For X, Y indiscernible sets take $\overline{a} \in X^k$, $\overline{b} \in Y^l$ such that $\overline{a} \not\mathrel{\unicode{x2AEB}} \overline{b}$ where $k + l$ is minimal. There is $x_0 \in Y \cap \operatorname{acl}(\overline{a})$ (look at $\operatorname{tp}\left(\overline{b}/\overline{a}\right)$ this is not Y^l so some element of Y^l is in acl of $\overline{a}$). Then $\overline{a} \not\mathrel{\unicode{x2AEB}} b_0$ so $l = 1$, and $k = 1$ by symmetry. $\square$

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[1] There is an extra step to get infinite rank which would happen at the end, but we won't have time to treat this.

Note that if $\overline{b} \in Y^k$ ($x_i \neq b_j$ for $i \neq j$) then $\operatorname{rank}\left(\overline{b}\right) = k$. If $a \in X$, $\operatorname{rank}(a) = 1$ so $\operatorname{rank}(\operatorname{acl}(a)) = 1$ so $|\operatorname{acl}(a) \cap Y| \leq 1$. This means $|\operatorname{acl}(a) \cap Y| = 1$ which gives us a definable bijection.

Now the following is a consequence of uniqueness. Let $X_1 \dots X_n$ be strictly minimal and assume (X_i, X_j) are orthogonal for $i \neq j$. Then for $\overline{a_i} \in X_i^{k_i}$

$$\operatorname{rank}\left(\overline{a_1}^n \wedge \dots \wedge \overline{a_n}\right) = \sum \operatorname{rank}\left(\overline{a_i}\right) \ .$$

1.2.2. *Back to the proof.* Write $\operatorname{rank}(M) = n$ and assume M is indivisible. There is $\varphi\left(x, \overline{b}\right)$ of rank $n-1$. Write $q = \operatorname{tp}\left(\overline{b}\right)$ We can assume

- $\varphi\left(\overline{x}, \overline{y}\right)$ is normalized
- $\varphi\left(x, \overline{b}\right)$ is indivisible for $\overline{b} \models q$.
- $\overline{b} \neq \overline{b'}$ implies $\varphi\left(X, \overline{b}\right) \neq \varphi\left(x, \overline{b'}\right)$ (taking $\overline{b} \in M^q$).

Then the goal is to show $\operatorname{rank}\left(\overline{b}\right) = 1$.

Let $F = \operatorname{tp}\left(\overline{b}\right)$ definable set. Assume $\operatorname{rank}(f) \geq 2$. Let $I\left(\overline{d}\right) \subseteq F$ be a strongly minimal definable subset. Let $H\left(\overline{d}\right)$ be the associated strictly minimal definable subset. We can assume $\overline{d} \in \operatorname{acl}(\emptyset)$, $q = \operatorname{tp}\left(\overline{d}\right)$.

Claim 1 (Main claim)**.** If $\overline{d_1} \downarrow \overline{d_2}$, then $I\left(\overline{d_1}\right) \Delta I\left(\overline{d_2}\right)$ is finite.

Proof. Assume not, then $I\left(\overline{d_1}\right) \cap I\left(\overline{d_2}\right)$ is finite. Take $\overline{d_1}, \dots, \overline{d_N}$ indiscernible independent N large.

Let $e = d_1 \wedge \dots \wedge d_N$. Let

$$Q = \{x \in M \mid \operatorname{rank}(x/e) = n\} \ .$$

Then Q is transitive over e since M is indivisible.

For a given i,

$$\bigcup_{\overline{b} \in I\left(\overline{d_i}\right)} \varphi\left(x, \overline{b}\right)$$

has rank n. so for all $x_0 \in Q$, for all i there is $\overline{b} \in I\overline{d_i}$ such that $\varphi\left(x_0, \overline{b}\right)$ holds. □

We work over e.

Claim 2. If $a \in Q$ and $\varphi\left(a, \overline{b}\right)$ holds with $\overline{b} \in I\left(\overline{d_i}\right)$ for some i. Then $\overline{b} \in \operatorname{acl}(a)$.

Proof. Otherwise a is in almost all $\varphi\left(x, \overline{b'}\right)$ for $\overline{b'} \in I(d_i)$. But then for any two, we can intersect them and

$$\operatorname{rank}\left(\varphi\left(x, \overline{b}\right) \cap \varphi\left(x, \overline{b'}\right)\right) = n-1$$

which means

$$\operatorname{rank}\left(\varphi\left(x, \overline{b}\right) \Delta \varphi\left(x, \overline{b'}\right)\right) < n-1$$

which contradicts normalization. Now we have two cases:

case 1. $H\left(\overline{d_1}\right) \dots H\left(\overline{d_N}\right)$ are orthogonal.

case 2. There is a unique $\emptyset$-definable set between $H\left(\overline{d_i}\right)$ and $H\left(\overline{d_j}\right)$.

□

Assume case 1. Then for any $a \in Q$, $\operatorname{acl}(a) \cap H\left(\overline{d_i}\right) \neq 0$ for all i so $\operatorname{rank}(a) \geq N > n$ which is a contradiction.

Assume case 2..

Claim 3. $\operatorname{rank}\left(\operatorname{acl}(a) \cap \left(H\left(\overline{d_1}\right) \cup \dots \cup H\left(\overline{d_N}\right)\right)\right) \geq N$.

Proof. For $\overline{b_1} \in I\left(\overline{d_1}\right)$ and $\overline{b_2} \in I\left(\overline{d_2}\right)$ we have

$$\operatorname{rank}\left(\varphi\left(x, \overline{b_1}\right) \cap \varphi\left(x, \overline{b_2}\right)\right) < n - 1$$

so if $a \in \varphi\left(x, \overline{b_1}\right) \cap \varphi\left(x, \overline{b_2}\right)$ then

$$\operatorname{rank}\left(a / \overline{b_1}\right) = n - 1 \qquad \operatorname{rank}\left(a / \overline{b_1 b_2}\right) \leq n - 2$$

so

$$\operatorname{rank}\left(\overline{b_2} / \overline{b_1}\right) \geq 1$$

so

$$\overline{b_2} \notin \operatorname{acl}\left(\overline{b_1}\right) \ .$$

So we got the same contradiction. □

So if we take for each i, $\overline{b_i} \in I\left(d_i\right)$ for each i then $\varphi\left(a, \overline{b_i}\right)$ holds

$$\operatorname{rank}\left(\overline{b_1}^n \wedge \ldots \wedge \overline{B_N}\right) \geq N$$

so $\operatorname{rank}(a) \geq N$ which is a contradiction.

This being done, normalize the family $I\left(\bar{b}\right)$ into $I^*\left(\bar{b}\right)$. Then

$$I^*\left(\overline{b_1}\right) = I^*\left(\overline{b_2}\right)$$

for $\overline{b_1} \perp \overline{b_2}$ so

$$I^*\left(\overline{b_1}\right) = I^*\left(\overline{b_2}\right)$$

for all $\overline{b_1}, \overline{b_2} \models q$ so $\operatorname{rank} F = 1$.

By the proof, F coordinatizes M.