LECTURE 4
MATH 229

LECTURE: PROFESSOR PIERRE SIMON
NOTES: JACKSON VAN DYKE

The title of this lecture is

a day at the zoo: examples.

This is typically a subject where there are very few general theorems, and one tests a technique by running through examples, so it is important to know them.

1. Stable structures

Consider the structure $X_0 = (X, =)$ where X is some countable set. $\mathrm{Aut}(X_0)$ is the full symmetric group $\mathrm{Sym}(X)$. One thing we can do is try to build finite covers of X_0. A finite cover[1] is an ω-categorical structure M with a projection

$$\begin{array}{c} M \\ \downarrow^{\pi} \\ X_0 \simeq M/E \end{array}$$

where E is a definable equivalence relation on M with finite bounded classes. The fact that the quotient M/E is exactly X_0 means there is no extra induced structure, i.e. there are no extra 0-definable sets. Equivalently, the canonical map $\mathrm{Aut}(\pi) : \mathrm{Aut}(M) \to \mathrm{Aut}(X_0)$ is surjective.

Example 1. A 2-cover is a finite cover where the equivalence classes have size 2. An immediate example is given by the disjoint union $M = X_0 \times \{0\} \cup X_0 \times \{1\}$ with the canonical projection. The automorphism group $\mathrm{Aut}(M) = \mathrm{Aut}(X_0) \times \mathbb{Z}/2\mathbb{Z}$.

Example 2. Another 2-cover of X_0 is given by putting 2 points above every points of X_0. So we have the equivalence relation mapping to X_0, but not the equivalence relation giving us the two copies. This is a reduct of the first one. So now the automorphism group is much bigger. In particular it is the wreath product $\mathrm{Aut}(X_0) \wr \mathbb{Z}/2\mathbb{Z}$.

Exercise 1 (*)**.** Prove these are the only two transitive 2-covers.

Example 3. Now we construct 4-covers. If we put 4 points over every point of X_0, with the following arrows:

$$\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] & & \cdot \arrow[dl] \\ & \cdot \arrow[ul] & \end{tikzcd} \tag{1}$$

Date: January 31, 2019.

[1]This same definition holds for any structure.

so the fibers are independent and the automorphism group is again the wreath product $\operatorname{Aut}(X_0) \wr \mathbb{Z}/4\mathbb{Z}$, i.e. because of the arrows the only automorphisms of the fibers are rotations.

Now we could of course trivialize everything as with the 2-covers and get a direct product, or we could try to do something in between. I.e. we can also add an equivalence relation which has 2-classes that picks out 2 opposite points. In particular, we can take the top and bottom points in each fiber to be equivalent to the top and bottom points in every other fiber. Of course if we just look at each fiber, we just have this $\mathbb{Z}/4\mathbb{Z}$ acting on it, but now the fibers aren't completely independent. If we rotate one fiber by 1, it doesn't force us to turn the other fibers exactly by 1, but it does force us to either rotate it by 1 or 3. So the automorphism group is something in between $\operatorname{Aut}(X_0) \times \mathbb{Z}/4\mathbb{Z}$ and $\operatorname{Aut}(X_0) \wr \mathbb{Z}/4\mathbb{Z}$.

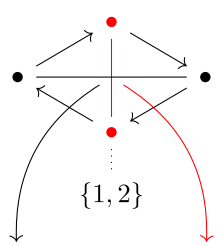
So in the 2-cover case, we couldn't sort of mix the situations, we only had these two. But now we have discovered a 4-cover which is sort of in between the two extremes. This turns out to somehow be as complicated as it gets.

Now a much more fun and complicated situation is the following sort of example:

Definition 1. The 2-Grassmannian of X_0 is the set of 2-element subsets of X_0 with the induced structure.

The idea is that each point is a 2-element subset, and then there is an edge whenever two of these subsets intersect.

Example 4. There is an interesting 4-cover of this. Above every point we can put 4 points, as in (1), except now we have 1 equivalence relation with classes of size 2, where each class is mapped to an elements in X_0. The picture is like this:



where the lines without arrows denote the 2-classes, and the color of the arrow mapping to X_0 indicates the corresponding class.

Proposition 1. *This structure is not interpretable in X_0.*

This is hard to prove, but we can at least see it isn't obviously interpretable. Say we introduce 4 parameters a, b, c, and d and then the points in a fiber are labelled by the pair of element of X_0 and one of the new parameters. So maybe we identify $((1,2), a)$ with $((2,1), c)$ But then we don't know what to send to 1, and what to send to 2 because of the orientation of the arrows. But then we have to check that no other identification works either. Note however that it is interpretable in DLO.

Example 5. Let M be the set of 4-element subsets of X_0 equipped with a partition into two sets of size 2. This is interpretable in X_0. Also, M is ω-categorical, but cannot be made homogeneous in a finite relational language. I.e. it is not interdefinable with a structure which is homogeneous in a finite relational language.

Exercise 2. Show that this is true.

Remark 1. We know that a structure being ω-categorical means there are only finitely many n-types for any n. So what is the extra property we get from being homogeneous in a finite relational language which is not syntactic? As it turns out, an ω-categorical structure M is interdefinable with a structure homogeneous in a finite relational language of maximal arity k iff n-types are determined by their k-subtypes. In other words

$$\bigwedge_{i_1<\ldots<i_k<n} \operatorname{tp}\left(a_{i_1}\cdots a_{i_k}\right) \vdash \operatorname{tp}\left(a_1\cdots a_n\right) \ .$$

This is surprising, however if we built the same M starting with DLO instead of X_0, then we get a finitely homogeneous structure. I.e. once we have an order, everything trivializes. This gives an example of a reduct of a finitely homogeneous structure which is not finitely homogeneous.

Theorem 1 (Lachlan)**.** *Every fintiely homogeneous stable structure (in particular finite covers of X_0) is interpretable in DLO.*

Remark 2. Note that DLO is not stable.

2. Orders

Linear orders are easy to deal with. DLO is the unique linear order which is homogeneous in the language consisting of just $\{\leq\}$. Partial orders become more complicated. We might ask the same question of which partial orders are homogeneous in $\{\leq\}$ (the language with just a partial order symbol). There is a classification of this due to Schmerl:

(1) DLO
(2) Fraissé limit of all partial orders. (This exists and is unique as a consequence of partial orders having amalgamation.)
(3) X_0
(4) Dense chain of antichains. This is where we put an anti-chain of size $\leq \aleph_0$ above every point of DLO.
(5) Disjoint union of chain. So an anti-chain of chains of a fixed size $\leq \aleph_0$.

Exercise 3 (*)**.** Show that any ω-categorical structure has an expansion by adding a linear order in such a way that it remains ω-categorical.

3. Graphs

The classification of homogeneous[2] graphs is due to Lachlan-Woodrow:

(1) Random graph
(2) Empty graph/complete graph
(3) K_n-free random graphs and complements of these.
(4) Disjoint union of cliques of the same size (includes some finite graphs).
(5) C_5 (Note C_4 is the complement of the equivalence relation with two classes of size 2, so included in the previous one and C_6 is not homogeneous)
(6) $K_3 \otimes K_3 = \{(a,b) \mid 1 \leq a \leq 3, 1 \leq b \leq 3\}$ where we put an edge between (a,b) and (c,d) iff $a \neq c$ and $b \neq d$.

[2]In the language of graphs.

4. Metric spaces

In this course, a metric space is presented in a binary language where we have one binary relation for every possible distance. Specifically the language

$$\{d_r(x,y) \mid r \in \mathbb{R}_{r\geq 0}\}$$

such that:

(i) $d_r(x,y) \wedge d_s(y,z) \wedge d_t(x,z) \implies r \leq s+t$,
(ii) $d_r(x,y) \iff d_r(y,x)$,
(iii) $d_0(x,y) \iff x=y$,
(iv) $d_r(x,y) \wedge d_s(x,y) \implies r=s$,
(v) $\forall x,y \bigwedge_{r\in\mathbb{R}_{\geq 0}} d_r(x,y)$.

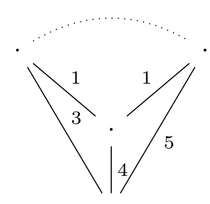
For $S \subseteq \mathbb{R}_{\geq 0}$, an S-metric space is the same but with only $\{d_r(x,y) \mid r \in S\}$.

Example 6. For $S = \{0,1\}$, this is just equality. For $S = \{0,1,2\}$, this is just a graph where edges are given by distance 1, nonedges are given by distance 2, and 0 is equality.

Question 1. Given S, does the class of S-metric spaces have amalgamation?

For $S = \mathbb{Q}$, the answer is yes, and the Fraissé limit (which is not ω-categorical) is called the rational Urysohn space. If S is closed under $+$, then the same.

Example 7. For $S = \{0,1,3,4,5\}$ there is no amalgamation:



since the dotted path cannot be filled in with anything satisfying the triangle inequality.