LECTURE 6: EPPA
MATH 229

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Today we start chapter 2: combinatorics.[1]

1. Introduction

Today we will talk about the extension property for partial automorphisms (EPPA). Consider some Fraïssé class $\mathcal{C}$ (finite relational). We know how to associate this to a homogeneous structure $\mathcal{M}$. Then a natural question is if we can find elements in $\mathcal{C}$ which are already homogeneous, or at least have many automorphisms.

Definition 1. We say $\mathcal{M}$ is smoothly approximable if for every $A \in \mathcal{C}$ there is homogeneous $B \in \mathcal{C}$ for which $A \hookrightarrow B$.

This is the best possible situation, but it rarely happens.

Theorem 1. *The smoothly approximable finitely homogeneous structures are precisely the stable finitely-homogeneous structures.*[2]

Definition 2. $\mathcal{C}$ has EPPA if for every $A \in \mathcal{C}$ there is $B \in \mathcal{C}$ such that $A \hookrightarrow B$ and every partial automorphism of A extends to an automorphism of B.

When this is true, B is called an EPPA witness for A. This turns out to be satisfied very often.

Remark 1. Note that EPPA and the joint embedding property (JEP) together imply the amalgamation property (AP).

Proof. Consider some B and C overlapping in A. Then JEP implies that there is some D_0 such that B and C embed and ρ identifies A living inside of these. Then this embeds in some EPPA witness D_1 of D_0, so ρ extends to an automorphism σ. Then D_1 amalgamates $\sigma(B)$ and C over A. □

2. Main theorems

First we see some examples.

Example 1. A class of linearly ordered structures cannot have EPPA.

Example 2. $(M, =)$ has EPPA. Consider some partial bijection, then take the set A itself as an EPPA witness, and of course you can extend any injection from $A \to A$ to a bijection $A \to A$.

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[1] Chapter 3 will be automorphism groups.

[2] Essentially these are finite covers of things like $(M^n, =)$. Recall the 'zoo' lecture.

Example 3. Consider the class of bipartite graphs (G, U, V, R). Consider some partial automorphism of U. First we extend this to a bijection of U, and expand V to V' as follows. For every $U_0 \subseteq I$, extend V to V' by adding $|V|$ many points related exactly to U_0. So we have somehow symmetrically expanded V. This will be motivation for the proof of the next theorem.

Theorem 2. *The class of finite graphs has EPPA.*

Proof of Hubička, Konečný, and Nešetřil. Let $A = (V(A), E(A))$ be a finite graph. We want to send A into some very symmetric thing which somehow contains every possible type over A. Define the EPPA witness B as follows. Define

$$V(B) = \{(v, f_v) \mid v \in V(A), f_v : V(A) \setminus \{v\} \to \{0,1\}\}$$

and let $((v, f_v)) \in E(B)$ iff $v \neq w$, and $f_v(w) \neq f_w(v)$. B has two types of automorphisms that we're concerned over. First, any bijection φ of A extends canonically to an automorphism $\tilde{\varphi}$ of B:

$$\tilde{\varphi} : (v, f_v) \mapsto \left(\varphi(v), f_v \circ \varphi^{-1}\right) .$$

The second type is as follows. For any pair $(v, w) \in V(A)^2$, where $v \neq w$, we get $\theta_{v,w} \in \operatorname{Aut}(B)$ which maps $\theta_{v,w} : (v, f_v) \mapsto (v, f_v')$ where f_v' is defined as

$$\begin{cases} f_v'(w) = 1 - f_v(w) & \\ f_v'(x) = f_v(x) & x \neq w \end{cases}$$

and

$$(w, f_w) \mapsto (w, f_x') \qquad (x, f_x) \mapsto (x, f_x), x \neq \{v, w\} .$$

Now we need to embed $i : A \hookrightarrow B$ as follows. Send v to (v, f_v) where we choose f_v inductively. Each each step, when we decide if we want an edge, just choose the function to be the same or different correspondingly.

Now we extend partial automorphisms. Let $\rho : A \dashrightarrow A$ be a partial automorphism. First extend ρ to a bijection $f = \hat{\rho}$ of A. Consider the automorphism $\tilde{f}$ of B. $\tilde{f}$ and ρ coincide on the first coordinate, wherever ρ is defined inside B. The reason $\tilde{f}$ might not extend ρ is because they might not agree on the second coordinate. Write:

$$\rho : (v, f_v) \mapsto (w, f_w) \qquad (v, f_v) \mapsto (w', f_{w'}) .$$

So look at $v \in \operatorname{dom} \rho$, and let $i(v) = (v, f_v)$, and $(\rho(v), g) = i(\rho(v))$. Then $(\rho(v), h) = \tilde{f}((v, f_v))$. If for some x, $g(x) \neq h(x)$, compose $\tilde{f}$ with $\theta_{\rho(v),x}$. But this θ might create a mistake if $x \in \operatorname{im} \rho$. If we do this inductively, we need to see that we are not going to change it again, and compose by $\theta_{x,\rho(v)}$ later on. But the only way this can happen, is if $x \in \operatorname{im} \rho$, say $x = \rho(y)$, but then since ρ and $\tilde{f}$ are both automorphisms, this will not be a problem. □

Theorem 3. *The class of finite K_n-free graphs has EPPA.*

Proof. Let A be a finite K_n-free graph and let B_0 be an EPPA witness for A as a graph. This might have K_ns so we want to lift this to something which doesn't have them. For every point of B_0, we will look at all of the K_ns that contain that point, and assign a number to each of these. For every pair, we will look at their lift, and decide if they have an edge between them. They will only have an edge if they disagree on all of the triangles they are both in. In particular, this will ensure there are no K_ns.

We now do this explicitly. Define B to consist of

$$V(B) = \{(v, t_v) \mid v \in V_0\}$$

where t_v is a function from the set of K_ns containing v to $\{0, \cdots, n-1\}$. Then (v, t_v) and (w, t_w) have an edge iff $(v, w) \in E(B_0)$, and $t_v(c) \neq t_w(c)$ for every c containing v and w. Write $\pi : B \to B_0$ for the projection. Note that, as we would hope, B is a K_n-free graph. If $C \subseteq B$ is a K_n, then $\pi(c)$ is a K_n in B_0, and the value on $\pi(c)$ of points in C must all be distinct, but this is impossible.

To be continued...

$\square$