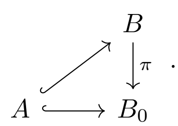
# LECTURE 7
# MATH 229

LECTURE: PIERRE SIMON
NOTES: JACKSON VAN DYKE

## 1. EPPA for $K_n$-free graphs

Recall we were proving that $K_n$-free graphs have EPPA.

*Proof.* Let $A$ be a $K_n$-free graph, and let $B$ be the EPPA witness of $A$ as a general graph. Then we want to build some $B$ by lifting $B_0$ in such a way that it is $K_n$-free. Then we need $A$ to embed into $B$, and automorphisms of $B_0$ to extend to $B$:




Define $B$ as follows:

$$V(B) = \{(v, t_r) \mid v \in B_0\}$$

where $t_r$ is a function assigning a value in $\{0, \cdots, n-2\}$ to every $n$-clique containing $v$. Then we have an edge $((v, t_v), (w, t_w)) \in E(B)$ iff $(v, w) \in E(B_0)$ and for any clique $c$ containing $v$, we have $f_v(c) \neq f_w(c)$. By construction, $B$ has no $n$-cliques. Note that we can embed $A$ into $B$ (lifting the embedding $A \hookrightarrow B_0$) by assigning values to cliques in an injective way. The point is that every clique of $B_0$ has at most $n-1$ points in $A$.

The automorphisms of $B$ are as follows. First we can lift every automorphism of $B_0$ in a natural way. To do this, we send:

$$\tilde{\varphi} : (v, t_v) \mapsto \left(\varphi(v), t_v \circ \varphi^{-1}\right) .$$

Another kind of automorphism is as follows. For any clique $c$ in $B_0$ and $\sigma \in \operatorname{Sym}(\{0, \cdots, n-2\})$ we have $\theta_{c,\sigma} \in \operatorname{Aut}(B)$ such that

$$\theta_{c,\sigma} : (v, t_v) \to (v, t'_v)$$

where $t'_v$ sends

$$t'_v : \begin{cases} k \mapsto t_v(k) & k \neq c \\ c \mapsto \sigma \circ t_v(c) \end{cases} .$$

Now we need to show that every automorphism of $A$ extends to one of $B$. Let $f : A \dashrightarrow A$. We already know $f$ extends to $\varphi \in \operatorname{Aut}(B_0)$ which gives $\tilde{\varphi} \in \operatorname{Aut}(B)$ which agrees with $f$ on the projection to $B_0$. For any clique $c$ of $B_0$, the image of $f$ (seen as $f : B \dashrightarrow B$) has at most $n-1$ points $(v, t_v)$ where $t_v$ is defined on $c$ and the values are all different. Similarly, to any such $\varphi$, the points $(w, t_w)$ in $\tilde{\varphi}(\operatorname{dom} f)$

*Date*: February 12, 2019.

take different values on $c$. Hence there is some $\sigma \in \mathrm{Sym}(\{0, \cdots, n-2\})$ such that $\theta_{c,\sigma} \circ \tilde{\varphi}$ and $f$ agree on the values at $c$. □

**Exercise 1.** Prove EPPA for 3-hypergraphs.[1] [Hint: take the valuations to give value in $\{0, 1\}$, and put a hyperedge if the sum of the values is odd/even.]

## 2. EPPA for metric spaces

Recall that for us, a metric space is a structure in a binary relational language where we have one binary relation for every possible distance. The general idea for proving EPPA is to regard it as a colored graph with edges between every two points. Assume we can find an EPPA witness $B_0$ for this as a colored graph.[2] Now we want to complete this to a metric space. One obstruction to completing $B_0$ is having a bad cycle, i.e. a cycle where one edge has distance larger than the sum of the others in the cycle. As it turns out we have the following:

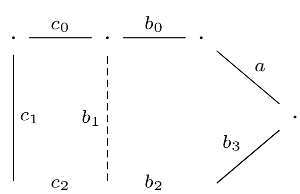
**Claim 1.** The only obstruction to completing a metric space is the existence of bad cycles.

*Proof.* For any $a, b \in B$, $a \neq b$ then set $\mathrm{dist}(a, b)$ to be

$$\min \left\{ \sum \mathrm{dist}(c_i, c_{i+1}) \mid a = c_0, c_1 \cdots, c_n = b; \mathrm{dist}(c, c+1) \text{ is defined in } B \right\} .$$

If this set is empty, i.e. there is no path between them, set $\mathrm{dist}(a, b)$ to be some large fixed distance.

The only thing to check is that we are not creating any triangles which fail the triangle inequality. It is enough to check that when we add one such distance, we have not created a bad cycle. Assume we have a bad cycle $\bar{b}$. The new edge cannot be the large edge of the cycle by construction, because if $a > \sum b_i$, then $a$ is defined as the minimum of all such things, so we have a contradiction. Now let the new value be one of the small edges of a bad cycle, say $b_1$. By definition, there exists a path $\bar{c}$ such that the sum of the $c_i$ is equal to the new edge, but now the concatenation of the bad cycle without $b_1$ and the path $\bar{c}$ is a bad cycle, so we must have had one to begin with, which is a contradiction. The picture is:




Note that all automorphisms of $B$ are preserved. □

This is saying that having a partial metric space is not something to be concerned about. So to prove EPPA for metric spaces, it is enough to construct an EPPA witness as a partial metric space (with no bad cycles). Now we want to lift this in the same way as we did for $K_n$ free graphs. There are two issues to deal with. The first is that we might have bad cycles of arbitrarily large size. This turns out to not be so bad, since we won't add any new distances until we complete it at the end.

[1] Or just for $n$-hypergraphs.

[2] The idea is to first give valuations for every edge, then treat possible multiple colors like we treated the cliques in the $K_n$ free case.

The other problem is that the argument we had before doesn't work exactly. If we lift and try to just break up the cycles by insisting that their assignment don't match, we can't detect if cycles are bad because somehow these things will only disagree in pairs. The proof that does work is the following.

*Proof of EPPA for metric spaces.* Let $A$ be a metric space. Take $B_0$ an EPPA witness for $A$ as a colored graph. Now we want to build $B$ which projects to $B_0$ such that $A$ embeds into it, where $B$ has no bad cycle. We restrict to the minimal bad cycles, i.e. it has no internal edges and doesn't repeat a vertex. Explicitly define $B$ by:

$$V(B) = \{(v, t_v) \mid v \in V(B_0)\}$$

where $t_v$ gives a value in $\{0, 1\}$ to every cycle $\{c_0, \cdots, c_{n-1}\}$ containing $v$. I.e. for every enumeration of the cycle it assign either 0 or 1.

Now define $\operatorname{dist}((v, t_v), (w, t_w)) = d$ iff we have that $\operatorname{dist}(v, w) = d$, and for every induced cycle $\overline{c} = \{c_0, \cdots, c_{n-1}\}$ containing $v$ $w$ we have one of the two cases:

- if there exists $i < n - 1$ such that
$$\{v, w\} = \{c_i, c_{i+1}\}$$
then this implies $t_v(\overline{c}) = t_w(\overline{c})$
- if $\{v, w\} = \{c_0, c_{n-1}\}$ then this implies $t_v(\overline{c}) \neq t_w(\overline{c})$.

The idea is that for each bad cycle, we follow one copy of it in $B$, and when we reach the end we jump over to the other copy, follow this, and then close it up.

First we need to check that $A$ embeds in $B$. We know $A$ can contain at most 2 points from any bad cycle, which must be consecutive since every pair of points in $A$ has a distance. Then we can just embed $A$ by a 'greedy' algorithm.

Now notice that the minimum size of a bad cycle increases when we pass from $B_0$ to $B$. This is sufficient because of the following. Since we know the values in $A$, there is a maximal size a bad cycle can have.[3] Consider some bad cycle in $B$. First assume it maps injectively, it means we had a bad cycle to begin with, except there are possible extra edges on the interior. But if this is the case, it is not induced, which means there is an induced bad cycle inside, so it is smaller. If it is already induced, then this contradicts the construction. Now if it is not injective, this means we are somehow collapsing two points, in which case we obtain two new cycles, but then the same long edge creates a smaller bad cycle as well.

As usual, the automorphisms of $B_0$ lift to automorphisms of $B$. Then for any bad cycle $\overline{c}$ we can flip $0 \leftrightarrow 1$ on $\overline{c}$. For any partial automorphism of $A$, any bad cycle intersects $A$ in at most two points, and the usual argument holds that this extends to $B$ by flipping 0s and 1s as needed. □

[3] This is something like $(\max / \min) + 1$.