LECTURE 10
MATH 242

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1. Classical mechanics

Recall we left off explaining some classical mechanics. Consider some Lagrangian function $L : TX \to \mathbb{R}$. The idea is that the trajectory of a particle is described by a path

$$\gamma : [a, b] \to X$$

which satisfies some conditions. The action is defined to be

$$A(\gamma) = \int_a^b L(\gamma(t), \gamma'(t))\, dt \ ,$$

and then γ should be a critical point of A subject to the boundary conditions given by fixed values of $\gamma(a)$ and $\gamma(b)$. In particular, γ is a critical point of A iff in local coordinates $x_1, \cdots, x_n$ on X, $v_1, \cdots, v_n$ on TX γ satisfies the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial v_i} \tag{1}$$

for all $i = 1, \cdots, n$.

Example 1. Let $X = \mathbb{R}^n$ and $L(x, v) = |v|^2/2 - f(x)$. Then for a particle of mass 1, the first term is the kinetic energy, and the second term is the potential energy. In this case, the equations (1) look like:

$$-\frac{\partial f}{\partial x_i} = \frac{d}{dt} v_i \ .$$

So with respect to time, the velocity is being pulled in the direction in which potential energy is getting smaller.

Now we want to see that this is equivalent to the Hamiltonian version.[1] In local coordinates as above, define

$$y_i = \frac{\partial L}{\partial v_i} \ .$$

The idea is that these are supposed to be coordinates of T^*X. The coordinate free version of this is defining a map

$$\mathcal{L} : TX \to T^*X \ .$$

Date: February 21, 2019.

[1] At least in nice cases.

At a point $(x, v) \in TX$, $\mathcal{L}(x, v)$ is defined to be the derivative of L in the fiber direction, i.e. a map

$$d_{x,v}\left(L|_{T_xX}\right) : T_vT_xX = T_xX \to \mathbb{R} \ .$$

This is called the Legendre transform. So these y_i are the coordinates on the cotangent bundle, i.e. any cotangent vector can be written:

$$\sum_i y_i \, dx_i \ .$$

Note that $\mathcal{L} : TX \to T^*X$ isn't necessarily a bundle map or anything, it's just a smooth map. But in good cases[2] $\mathcal{L}$ is a diffeomorphism.

Example 2. Let g be a Riemannian metric. Then for

$$L(x, v) = \frac{1}{2} g(v, v) - f(x)$$

this $\mathcal{L}$ is the isomorphism $TX \simeq T^*X$ determined by g.

Henceforth assume we are in such a situation. So now we have an inverse map $\mathcal{L}^{-1} : T^*X \to TX$. Locally this looks like $v_i = G_i(x, y)$. Now define $H : T^*X \to \mathbb{R}$ by

$$H(x, y) = \sum_i y_i G_i(x, y) - L(x, G(x, y)) \ .$$

If we wanted to write this without coordinates the first term would just be pairing tangent and cotangent vectors.

Now comes the confusing part.

Claim 1. Trajectories of the Hamiltonian vector field X_H correspond to paths $\gamma : [a, b] \to X$ such that $(\gamma, \gamma') : [a, b] \to TX$ satisfies (1).

Proof. The tricky thing is that $\partial H/\partial x_k$ is differentiating while keeping x_j for $j \neq k$ and y_j for all j fixed. Then $\partial L/\partial x_i$ is differentiating while keeping x_j for $j \neq i$ and all v_j fixed. Now we can calculate:

$$\frac{\partial H}{\partial x_k} = \sum_i y_i \frac{\partial G_i(x, y)}{\partial x_k} - \frac{\partial L}{\partial x_k}(x, G(x, y)) - \sum_i \frac{\partial L}{\partial v_i} \frac{\partial G_i}{\partial x_k}$$

but the first and last terms cancel by definition of y_i, so this is just

$$\boxed{\frac{\partial H}{\partial x_k} = -\frac{\partial L}{\partial x_k}} \ .$$

Now we have to do the other one which gives us:

$$\frac{\partial H}{\partial y_k} = G_k(x, y) + \sum_i y_i \frac{\partial G_i}{\partial y_k} - \sum_i \frac{\partial L}{\partial v_i} \frac{\partial G_i}{\partial y_k}$$

and now the second two terms cancel, so

$$\boxed{\frac{\partial H}{\partial y_k} = G_k} \ .$$

[2]In particular if L is strictly convex and proper.

Hamilton's equations are:

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k} \qquad \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}$$

so the boxed equations give us

$$\frac{dx_k}{dt} = v_k \qquad \frac{d}{dt}\frac{\partial L}{\partial v_k} = \frac{dy_k}{dt} = \frac{\partial L}{\partial x_k}$$

and we are done. □

Then we have the following, which was our original reason for doing this:

Corollary 1. *On a Riemannian manifold (X, g), the Hamiltonian vector field X_H for $H(x, y) = |y|^2/2$ on the unit (co)tangent bundle generates the geodesic flow.*

Proof. Define $L : TX \to \mathbb{R}$ by $L(x, v) = |v|^2/2$, then the Legendre transform is the identification $TX \to T^*X$ given by g and $L \leftrightarrow H$. So then trajectories of X_H correspond to critical points of the action functional

$$A(\gamma) = \int_a^b \frac{1}{2} |\gamma'|^2 \, dt \ ,$$

which are geodesics. □

2. Group actions on symplectic manifolds

2.1. S^1 **action.** Consider an S^1 action on (M, ω), i.e. for each $\theta \in S^1$ we have a symplectomorphism $\varphi_\theta : (M, \omega) \to (M, \omega)$ such that $\varphi_{\theta_1+\theta_2} = \varphi_{\theta_1} \circ \varphi_{\theta_2}$. This is generated by the vector field

$$X = \left.\frac{d}{d\theta}\right|_{\theta=0} \varphi_\theta \ .$$

2.2. **Hamiltonian actions.**

Definition 1. An S^1 action is called *Hamiltonian* if X is a Hamiltonian vector field, i.e. $X = X_H$ for some $H : M \to \mathbb{R}$.

Remark 1. Since the Lie derivative $\mathcal{L}_X\omega = 0$ (because the action preserves ω), we have $d\iota_X\omega \equiv 0$, so $\iota_X\omega$ is a closed 1-form. Then we have that the action is Hamiltonian iff this 1-form is exact.

2.3. **Symplectic quotient.** Given a Hamiltonian S^1 action on (M, ω) generated by H, suppose $c \in \mathbb{R}$ is a regular value of H, and that S^1 acts freely on the level set $H^{-1}(c)$. Then we can define the symplectic quotient to be:[3]

$$M /\!\!/ S^1 = H^{-1}(c)/S^1 \ .$$

Lemma 1. *Let $\pi : H^{-1}(c) \to M /\!\!/ S^1$ denote the projection. Then there is a unique symplectic form $\hat{\omega}$ on $M /\!\!/ S^1$ such that $\pi^*\hat{\omega} = \omega|_{H^{-1}(c)}$.*

[3]Note that this depends on c, which is not reflected by this notation.

Proof. Let $p \in H^{-1}(c)$, and $v, w \in T_{[p]}\left(M /\!\!/ S^1\right)$. Take some $\tilde{v}, \tilde{w} \in T_p H^{-1}(c)$ such that $\pi_* \tilde{v} = v$ and $\pi_* \tilde{w} = w$. We can find these because $\pi_* : T_p H^{-1}(c) \to T_{[p]}\left(M /\!\!/ S^1\right)$ is surjective with kernel $T_p\left(S^1 \cdot p\right) = \mathbb{R}\{X_H\}$. Define

$$\hat{\omega}(v, w) = \omega(\tilde{v}, \tilde{w}) \ .$$

Note that if a symplectic form is defined at all it must be given by this formula. So now we need to check it is well-defined and symplectic. This does not depend on $\tilde{v}$ and $\tilde{w}$ because X_H generates the symplectic complement of $TH^{-1}(c)$. Now note that this depends only on $[p]$ and not p because ω is S^1-invariant. $\hat{\omega}$ is closed because $\pi^* \hat{\omega}$ is closed and π_* is surjective:

$$d\hat{\omega}\ (u, v, w) = \pi^*\, d\hat{\omega}\ (\tilde{u}, \tilde{v}, \tilde{w}) = \ d\left(\pi^* \hat{\omega}\right)\ (\tilde{u}, \tilde{v}, \tilde{w}) = 0 \ .$$

Finally this is nondegenerate since X_H generates the symplectic complement of $TH^{-1}(c)$. $\square$

Example 3 (Important)**.** Let $M = \mathbb{C}^{n+1}$ and

$$\omega = \omega_{\text{std}} = \sum_{i=1}^{n+1} dx_i\ dy_i \ .$$

Define $H : \mathbb{C}^{n+1} \to \mathbb{R}$ by

$$H(z_0, z_1, \cdots, z_n) = \pi \sum_{i=0}^{n} |z_i|^2 \ .$$

Then we claim that X_H generates an S^1 action on $\mathbb{C}^{n+1}$. Recall the equation is that $\omega(X_H, -) = \ dH$, so this says that

$$X_H = \sum_i \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i} \ .$$

In our particular case we have

$$X_H = \sum_{i=0}^{n} 2\pi \underbrace{\left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}\right)}_{-\partial/\partial\theta_i} \ .$$

The action is given by $\theta \in \mathbb{R}/\mathbb{Z}$ sending

$$(z_0, \cdots, z_n) \mapsto \left(e^{-2\pi\sqrt{-1}\theta} z_0, \cdots, e^{-2\pi\sqrt{-1}\theta} z_n\right) \ .$$

So now we need to choose a c to take the symplectic quotient. $c < 0$ would give us the empty set which isn't very interesting. $c = 0$ isn't regular but $c > 0$ is fine. Let's take $c = 1/\pi$. Then $H^{-1}(c) = S^{2n+1}$ and we get a symplectic form on $\mathbb{C}^{n+1} /\!\!/ S^1 = S^{2n+1}/S^1 = \mathbb{CP}^n$. As it turns out this form is invariant under the action of $\mathrm{U}(n+1)$, and on $T_{[1:0:\ldots:0]}\mathbb{CP}^n$ this agrees with the standard symplectic form sort of by construction. In fact, one way to specify this form would be to say that it is the unique 2-form such that it is the standard form on such a tangent space, and it is $\mathrm{U}(n+1)$ invariant. It maybe isn't obvious that such a 2-form exists, but this construction shows us that it does.

An explicit definition of the symplectic form on $\mathbb{CP}^n$ is as follows. Where $z_j \neq 0$,

$$\omega = \frac{i}{2}\partial\bar{\partial}\log\left(\frac{|z|^2}{|z_j|^2}\right) \ .$$

This is called the Fubini-Study form.

Remark 2. If $X \subset \mathbb{CP}^n$ is a smooth complex variety, then $\omega|_X$ is a symplectic form on X. This gives us a huge collection of symplectic manifolds.

Proof. $\omega|_X$ is closed because ω is closed to begin with. So we just gave to show non-degeneracy. ω satisfies the equation

$$\omega(v, w) = \langle Jv, w \rangle$$

where this is the inner product on $\mathbb{CP}^n$ induced by the standard metric on S^{2n+1}. This is true because it holds on the tangent space, and holds elsewhere by $U(n+1)$ symmetry. Now $\omega|_X$ is nondegenerate because if $v \in TX$ is nonzero, then $\omega(V, Jv) = \langle JV, JV \rangle \neq 0$. $\square$