LECTURE 11
MATH 242

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1. Group actions on Symplectic manifolds

1.1. **S^1-actions.** Recall we were dealing with the following situation. Consider a Hamiltonian S^1 action on (M, ω) generated by X_H where $H : M \to \mathbb{R}$. If $c \in \mathbb{R}$ is a regular value of H on which S^1 acts freely, then

$$H^{-1}(c) / S^1 = M /\!\!/ S^1$$

is the symplectic quotient.

Example 1. We did the example last time where $(M, \omega) = \left(\mathbb{C}^{n+1}, \omega_{\text{std}}\right)$ and $H = \pi |z|^2$. The corresponding symplectic quotient was $\mathbb{CP}^n$.

1.2. **Torus actions.** We write $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$.

Definition 1. A Hamiltonian $\mathbb{T}^k$ action on (M, ω) is a $\mathbb{T}^k$ action on (M, ω) together with a "moment map" $\mu : M \to \mathbb{R}^k = \left(\mathfrak{t}^k\right)^*$ such that:

1. if $v \in \mathbb{R}^k = \mathfrak{t}^k$, then the derivative of the $\mathbb{T}^k$ action in the direction of v is the Hamiltonian vector field of $H_v = \langle \mu(\cdot), V \rangle$, and
2. μ is invariant under the $\mathbb{T}^k$ action.

Remark 1. There is a more general version of this where instead of $\mathbb{T}^k$ we have a Lie group G. The first condition is the same, and the second condition says that it is G-equivariant where G acts by the adjoint action.

If $c \in \mathbb{R}^k$ is a regular value of μ such that $\mathbb{T}^k$ acts freely on $\mu^{-1}(c)$, then

$$\mu^{-1}(c) / \mathbb{T}^k = M /\!\!/ \mathbb{T}^k$$

is naturally a symplectic manifold. Note that

$$\dim\left(M /\!\!/ \mathbb{T}^k\right) = \dim(M) - 2k \ .$$

1.3. **Toric symplectic manifolds.** We get an interesting class of symplectic manifolds by considering the tori which give rise to a 0-dimensional symplectic quotient.

Definition 2. A (compact) *toric symplectic manifold* is a (compact) symplectic manifold $\left(M^{2n}, \omega\right)$ together with an effective Hamiltonian $\mathbb{T}^n$ action with moment map $\mu : M \to \mathbb{R}^n$.

These aren't so interesting from the point of view of symplectic quotients since they just give a point, but they are interesting examples of symplectic manifolds that come up a lot.

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Theorem 1 (Atiyah-Guillemin-Sternberg, Delzant)**.**

- *For a compact connected toric symplectic manifold, $\mu(M) \subset \mathbb{R}^k$ is a convex polytope. For each vertex, there are n adjacent edges, whose vectors, after rescaling, are in $\mathbb{Z}^n$ and give a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*
- *The map $M \mapsto \mu(M)$ gives a bijection between compact, connected, toric symplectic manifolds modulo isomorphism, and convex polytopes as above.*

Example 2. Let $M = \mathbb{CP}^n$ and $\mu : \mathbb{CP}^n \to \mathbb{R}^n$ be defined by

$$\mu(z_0 : \cdots : z_n) = \frac{\pi\left(|z_1|^2, \cdots, |z_n|^2\right)}{|z_0|^2 + \cdots + |z_n|^2} \ .$$

The action of $(\theta_1, \cdots, \theta_n) \in \mathbb{T}^n$ is given by

$$[z_0 : \cdots : z_n] \mapsto \left[z_0 : e^{2\pi i\theta_1} z_1 : \cdots : e^{2\pi i\theta_n} z_n\right] \ .$$

The image of μ is

$$\mu(\mathbb{CP}^n) = \left\{x \in \mathbb{R}^n \mid x_i \geq 0, \sum x_i = \pi\right\} \ .$$

For $n = 2$, this is just a right triangle with vertices at $(0,0)$, $(\pi, 0)$, and $(0, \pi)$. The inverse image of the origin is the point $[1 : 0 : 0]$. The inverse image of $(0, \pi)$ is $[0 : 0 : 1]$, and the inverse image of $(\pi, 0)$ is $[0 : 1 : 0]$. The inverse image of the point $(\pi/2, 1)$ is the circle of all points of the form $\left[1 : e^{i\theta} : 0\right]$. The inverse image of the whole bottom edge is the $\mathbb{CP}^1$ given by points of the form $[z_0 : z_1 : 0]$, the inverse image of the left edge is a $\mathbb{CP}^1$ given by points of the form $[z_0 : 0 : z_2]$, and the third edge is a $\mathbb{CP}^1$ given by points of the form $[0 : z_1 : z_2]$. The inverse image of an interior point is a 2-torus consisting of points which look like $\left[1 : e^{i\theta_1} r_1 : e^{i\theta_2} r_2\right]$ for fixed r_1 and r_2.

Fact 1. *These are Lagrangian tori.*

Now consider an inscribed similar triangle with right vertex at the origin. If we draw a similar triangle also with right angle at the origin, then the long edge Then the long edge (without endpoints) of this smaller triangle has inverse image a torus cross an interval. If we include the endpoints we get S^3. The inverse image of this entire triangle is B^4.

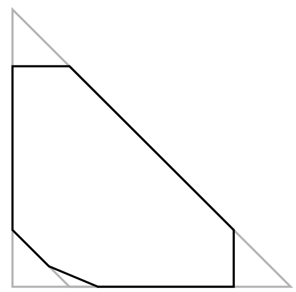
Another interesting thing to do is to remove this small triangle. This is still a perfectly legitimate polytope, so by the above theorem, this corresponds to some toric symplectic manifold.[1] In particular this corresponds to $\mathbb{CP}^2$ blown up at a point. This shorter diagonal edge is the exceptional divisor. The top diagonal edge is the line at ∞ in $\mathbb{CP}^2$, so if we delete this edge, this has inverse image $\mathbb{C}^2$ blown up at a point. This consists of the pairs (L, z) where L is a complex line through the origin in $\mathbb{C}^2$, and $z \in L$. For nonzero z this determines L uniquely, but if z is 0, L could be anything. So there is a projection from this to $\mathbb{C}^2$ which is a bijection away from the origin. But at the origin, the inverse image is $\mathbb{CP}^1$.

Remark 2 (Blow-ups in 4-dimensions). In algebraic geometry, a point is removed and replaced by $\mathbb{CP}^1$, which is called the exceptional divisor. In symplectic geometry, a 4-ball is removed and the boundary S^3 is collapsed by identifying fibers of the

[1] As it turns out, the generic way to turn a polytope into a symplectic manifold is by taking the polytope cross a torus, and then collapse the corners to points, and collapse the edges to circles.

Hopf fibration to the S^2. If you're a 4-manifold topologist, then this is a connect sum with $\overline{\mathbb{CP}^2}$.

We can continue to chop off corners as long as we have an integer basis, e.g. if we chop 4 corners off to get the following:



then this thing corresponds to $\mathbb{CP}^2$ blown up at 4-points. Each time we do it, we pay the price of losing a bit of symplectic volume. In the 4-dimensional case this is the same as the area of the polytope.

The rectangle in the plane corresponds to $S^2 \times S^2$ since the cartesian product of a polytope corresponds to the cartesian product of the symplectic manifolds.

Fact 2 (fun)**.** *In $\mathbb{CP}^n$, the Lagrangian n-torus given by the preimage of the center of $\mu(\mathbb{CP}^n)$:*

$$\mu^{-1}\left(\pi\left(\frac{1}{n},\cdots,\frac{1}{n}\right)\right)$$

is not displaceable. This can be proved using Lagrangian Floer homology.

It is also true that all other fibers $\mu^{-1}(x)$ (where x is in the interior of the moment map image) are displaceable (this is at least know for $n \leq 3$).

2. Almost complex structures

Let $\left(M^{2n},\omega\right)$ be a symplectic manifold. An almost complex structure on M is a bundle map $J : TM \to TM$ such that $J^2 = -1$. This makes TX into a complex vector bundle. We want to insist that these have some sort of compatibility with our symplectic form.

Definition 3. J is ω-compatible if the bundle map

$$TM \otimes TM \xrightarrow{\quad g \quad} \mathbb{R}$$

$$(v,w) \longmapsto \omega(v,Jw)$$

is a positive definite inner product on TM. I.e. g is a Riemannian metric on M.

First of all this means that g is symmetric, i.e. $g(v,w) = g(w,v)$. But this is equivalent to:

$$\omega(v,Jw) = \omega(w,Jv) = \omega\left(-J^2w,Jv\right) = \omega\left(Jv,J^2w\right)$$

so if we define $u = Jw$ this just says

$$\omega(v,u) = \omega(Jv,Ju)\ .$$

So this is equivalent to ω being invariant under the action of J. Then positive definite means $g(v,v) \geq 0$, with equality iff $v = 0$.

Proposition 1. *The set of ω-compatible almost complex structures on (M, ω) is contractible.*

Proof. It is enough to show that this is true on each fiber, i.e. we will show that for (V, ω) a symplectic vector space, the set of linear maps $J : V \to V$ such that $J^2 = -1$, $\omega(Ju, Jv) = \omega(u, v)$, and $\omega(u, Jv) \geq 0$ (with equality iff $v = 0$) is contractible. Call this set $\mathcal{J}(V, \omega)$.

This is sufficient because of the following. There is a fiber a bundle $E \to M$ such that the fiber over $p \in M$ is $\mathcal{J}\left(T_pM, \omega|_{T_pM}\right)$. An ω-compatible J is then just a section of this bundle. If all the fibers of this bundle are contractible, then the set of sections is contractible.

To see that $\mathcal{J}(V, \omega)$ is contractible, it is enough to show that the space $\mathcal{J}\left(\mathbb{R}^{2n}, \omega_{\text{std}}\right)$ is contractible. Let J_0 be multiplication by i on $\mathbb{R}^{2n} = \mathbb{C}^n$. Then notice that

$$\omega_{\text{std}}(u, v) = u^T J_0 v \ .$$

Now we can rewrite the set $\mathcal{J}\left(\mathbb{R}^{2n}, \omega_{\text{std}}\right)$ as:

$$\left\{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid J^2 = -1, J^T J_0 J = J_0, v^T J^T J_0 v \geq 0, = 0 \iff v = 0\right\} \ .$$

Next time we will prove a lemma that reduces the question to one concerning symmetric positive definite matrices. And then the space of such matrices is contractible since we can somehow deform all eigenvalues to 1. □