LECTURE 12
MATH 242

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1. Almost complex structures

1.1. **ω-compatible acs.** Recall we defined an almost-complex structure to be a bundle map $J : TM \to TM$ with $J^2 = -1$. Then J is ω-compatible if $g(u, v) := \omega(u, Jv)$ is a Riemannian metric. We were in the middle of the proof of the following:

Proposition 1. *The space of ω-compatible almost complex structures J is contractible.*

Continued proof. Recall it is sufficient to show that the set

$$\mathcal{J}\left(\mathbb{R}^{2n}, \omega_0\right) = \left\{J : \mathbb{R}^{2n} \circlearrowleft \mid J^2 = -1, (u, v) \mapsto \omega(u, Jv) \text{ is pos. def. inner prod.}\right\}$$

is contractible. We have the following lemma:

Lemma 1. *A linear map $J \in \mathcal{J}\left(\mathbb{R}^{2n}, \omega\right)$ iff $-J_0 J$ is symmetric, symplectic, and positive definite.*

Proof. ($\Longrightarrow$): By definition, $J \in \mathcal{J}\left(\mathbb{R}^{2n}, \omega\right)$ iff $J^2 = -1$, $\omega(u, Jv) = \omega(v, Ju)$, and $\omega(v, Jv) \geq 0$ (with equality iff $v = 0$). The second condition is equivalent to $u^T J_0 J v = -u^T J^T J_0 v$ which is equivalent to

$$J_0 J = -J^T J_0 \ . \tag{1}$$

So given these conditions we want to show that $-J_0 J$ is symmetric, symplectic, and positive definite. To see that it is symmetric, we can just check that

$$(-J_0 J)^T = -J^T J_0^T = J^T J_0 = -J_0 J$$

where we have used (1) in the last step. Recall that a matrix A is symplectic when $A^T J_0 A = J_0$. So for $A = -J_0 J$ we can write

$$(-J_0 J)^T J_0 (-J_0 J) = J^T J_0^T J_0 J_0 J = -J^T J_0^T J = J^T J_0 J \overset{1}{=} -J_0 J J = J_0 \ .$$

Positive definite is easy to check.

($\Longleftarrow$): This direction is effectively the same. □

Now the result follows because the space of symmetric, symplectic, and positive definite matrices is contractible. ■

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[1] by (1)

1.2. **ω-tame acs.**

Definition 1. An acs J is ω-tame if $\omega(v, Jv) \geq 0$ with equality iff $v = 0$.

Remark 1. J is ω-compatible iff J is ω-tame and J is symplectic, i.e. $\omega(Ju, Jv) = \omega(u, v)$.

Fact 1. *The space of ω-tame J is also contractible.*

1.3. **First results about acs.**

Remark 2. Since the space of ω-compatible J is contractible, TM has a distinguished structure of a complex vector bundle which is unique up to homotopy.

In particular, this implies that there are well-defined Chern classes $c_k(TM) \in H^2(M, \mathbb{Z})$ for $k = 0, \cdots, n$. Note that these $c_k(TM)$ are invariant under deformation of ω.

Remark 3 (Chern class review). If $L \to B$ is a complex line bundle over a CW-complex B, we can define the first Chern class, $c_1(L) \in H^2(B)$ to be the Euler class of the associated circle bundle.

If E is a rank k complex vector bundle, we can define

$$c_1(E) = c_1(\det E) = c_1\left(\wedge^k E\right)$$

to be the first Chern class of the determinant line bundle.

If J_0 and J_1 are ω-compatible acs on (M, ω), this implies that there exists an isomorphism of complex vector bundles:

$$\begin{array}{ccc} (TM, J_0) & \xrightarrow{\simeq} & (TM, J_1) \\ & \searrow \quad \swarrow & \\ & M & \end{array} .$$

1.4. **Difference between acs and complex structures.** Let M be a $2n$-dimensional real smooth manifold. An almost complex structure is a bundle map $J : TM \to TM$ such that $J^2 = -1$. Now a *complex manifold structure* is a maximal atlas of coordinate charts:

$$\varphi : U \xrightarrow{\simeq} V$$

(which are diffeomorphisms for U some open subset of M, and V some open subset of $\mathbb{C}^n$) such that the transition maps between open subsets of $\mathbb{C}^n$ are holomorphic. This makes M into an n-dimensional complex manifold.

A complex structure gives rise to an acs J given by multiplication by i. The converse is not true.

Definition 2. An almost complex structure $J : TM \circlearrowleft$ is integrable if it comes from a complex manifold structure as above.

Not every acs is integrable, but there is a theorem saying when they are. First consider the following. If J is an acs, it gives a decomposition

$$\wedge^k M \otimes \mathbb{C} = \bigoplus_{i+j=k} T^{i,j} M$$

where the pieces look something like

$$dz_{p_1} \wedge \cdots \wedge dz_{p_i} \wedge d\bar{z}_{q_1} \wedge \cdots \wedge d\bar{z}_{q_j} .$$

Now we define the Nijenhuis tensor

$$N : T^{1,0} \to T^{0,2} \ .$$

If α is a $(1,0)$-form (i.e. a section of $T^{1,0}$), define $N\alpha$ to be the projection to $T^{0,2}$ of $d\alpha$. Now we have to check this is a tensor:

$$N(f\alpha) = (d(f\alpha))^{0,2} = \cancel{(df \wedge \alpha)^{0,2}} + (f \wedge d\alpha)^{0,2} = fN\alpha$$

where $df = \partial f + \bar{\partial} f$. Now notice if J is integrable this is certainly 0, but in fact we have the following:

Theorem 1 (Newlander-Nirenberg)**.** *J is integrable iff $N \equiv 0$.*

Remark 4. If $\dim_{\mathbb{R}} M = 2$, then $N = 0$ automatically. So any acs is integrable in this case.

2. Holomorphic curves

These are also called pseudo-holomorphic curves, and J-holomorphic curves. Let (Σ, j) be Riemann surface (not necessarily compact). I.e. Σ is a one-dimensional complex manifold, and $j : TM \circlearrowleft$ is multiplication by i.

Remark 5. By the above remark, a Riemann surface is equivalent to a pair (Σ, j) where Σ is a real 2-manifold and $j : T\Sigma \circlearrowleft$ is an acs.

Definition 3. Given (M, ω, J) where J is ω-compatible/tame, a J-holomorphic map $(\Sigma, j) \to (M, \omega, J)$ is a smooth map $u : \Sigma \to M$ such that du is complex linear, i.e.

$$J \circ du = du \circ j \ .$$

2.1. Some basic features of these.

Note that if J is ω-tame, it determines a Riemannian metric g on M defined by

$$g(u, v) = \frac{1}{2}\left(\omega(u, Jv) + \omega(v, Ju)\right) \ .$$

If J is ω-compatible, then

$$g(u, v) = \omega(u, Jv) \ .$$

Now we consider areas of holomorphic maps. Recall that if (M, g) is any Riemannian manifold, and $u : \Sigma \to M$ is any smooth map, we can define the area to be

$$\operatorname{area}(u) = \int_\Sigma \sqrt{\det g\left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\right)} \, \left| dx^1 \cdots dx^k \right| \ .$$

Note that if Σ is a submanifold of M and u is the inclusion map, then this is the usual area.

Proposition 2. *If $u : (\Sigma, j) \to (M, J)$ is holomorphic, and J is ω-tame then*

$$\operatorname{area}(u) = \int_\Sigma u^* \omega \ .$$

Proof. We will check that the integrands agree pointwise. Let $p \in \Sigma$ and $v \in T_p\Sigma \setminus \{0\}$. Then (v, jv) is a basis for $T_p\Sigma$. We need to check that

$$\sqrt{\det \begin{pmatrix} g(du(v), du(v)) & g(du(v), du(jv)) \\ g(du(v), du(jv)) & g(du(jv), du(jv)) \end{pmatrix}} \ .$$

Let's rewrite this a bit neater by writing $w = du(v)$. Then $Jw = du(jv)$. So we need to show that

$$\sqrt{\det \begin{pmatrix} g(w,w) & g(w,Jw) \\ g(w,Jw) & g(Jw,Jw) \end{pmatrix}} = \omega(w,Jw) \ .$$

Now we can just calculate:

$$\begin{aligned} g(w,w) &= \omega(w,Jw) \\ g(w,Jw) &= \frac{1}{2}\left(\omega\left(w,J^2w\right) + \omega(Jw,Jw)\right) = 0 \\ g(Jw,Jw) &= \omega\left(Jw,J^2w\right) = \omega(Jw,-w) = \omega(w,Jw) \end{aligned}$$

which means

$$\sqrt{\det \begin{pmatrix} g(w,w) & g(w,Jw) \\ g(w,Jw) & g(Jw,Jw) \end{pmatrix}} = \sqrt{\omega(w,Jw)^2}$$

which is positive because J is ω-tame so we are done. □

Proposition 3. *Suppose that J is ω-compatible, $u : (\Sigma, j) \to (M, \omega, J)$ is holomorphic, and Σ is compact so that*

$$u_*[\Sigma] \in H_2(M)$$

is defined. Then u is area minimizing in its homology class. That is, if $v : \Sigma' \to U$ is a smooth map with

$$v_*[\Sigma'] = u_*[\Sigma] \in H_2(M) \ ,$$

then $\operatorname{area}(v) \geq \operatorname{area}(u)$.

Proof. We know that

$$\operatorname{area}(u) = \int_\Sigma u^*\omega = \int_\Sigma \langle u_*[\Sigma], \omega \rangle$$

so we need to show that

$$\operatorname{area}(v) \geq \int_\Sigma v^*\omega \ .$$

We will prove this pointwise. Let $p \in \Sigma'$, and let $\{X, Y\}$ be a basis for $T_p\Sigma'$. We want to show that

$$\sqrt{\det \begin{pmatrix} g(v_*X, v_*X) & g(v_*X, v_*Y) \\ g(v_*X, v_*Y) & g(v_*Y, v_*Y) \end{pmatrix}} \geq \omega(v_*X, v_*Y) \ .$$

WLOG assume $v_*X \neq 0$ (o/w we are done) and $g(v_*X, v_*X) = 1$. Since J is ω-compatible, we can choose a basis

$$\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$$

for $T_{V(p)}M$ such that $Je_i = f_i$, and

$$\begin{aligned} \omega(e_i, e_j) = \omega(f_i, f_j) &= 0 & \omega(e_i, f_j) &= \delta_{ij} \\ g(e_i, f_j) &= 0 & g(e_i, e_j) = g(f_i, f_j) &= \delta_{ij} \ . \end{aligned}$$

The idea is to choose a basis to look like $\mathbb{C}^n$. Write

$$v_*Y = \sum_{i=1}^n (a_ie_i + b_if_i) \,.$$

Now we can directly calculate

$$g(v_*X, v_*X) = 1$$
$$g(v_*X, v_*Y) = a_1$$
$$g(v_*Y, v_*Y) = \sum \left(a_i^2 + b_i^2\right)$$

so we have

$$\sqrt{\det(\cdots)} = \sqrt{-a_1^2 + \sum_{i=1}^{2} (a_i^2 + b_i^2)}$$

and

$$\omega(v_*X, v_*Y) = b_1$$

so we get the desired inequality. Note this implies the image of du_p is complex linear. $\square$