LECTURE 13
MATH 242

LECTURE: PROFESSOR MICHAEL HUTCHINGS
NOTES: JACKSON VAN DYKE

1. J-holomorphic curves

1.1. **Definitions.** Recall we were considering a symplectic manifold with an ω-compatible acs $\left(M^{2n}, \omega, J\right)$. Then a J-holomorphic map is $u : (\Sigma, j) \to (M, \omega, J)$ such that

$$J \circ du = du \circ j \; .$$

Then we saw that if Σ is compact, then

(1) $\operatorname{area}(u) = \langle u_* [\Sigma], [\omega] \rangle$
(2) u minimizes area among smooth maps $u' : \Sigma' \to M$ in the same homology class.

Definition 1. A J-holomorphic curve is a J-holomorphic map $u : (\Sigma, j) \to (M, \omega, J)$ modulo the equivalence relation that u is equivalent to $u' : (\Sigma', j') \to (M, \omega, J)$ if there exists a biholomorphic map

$$\varphi : (\Sigma, j) \xrightarrow{\cong} (\Sigma', j')$$

such that $u = u' \circ \varphi$.

Remark 1. An embedded J-holomorphic curve is determined by its image in M, and an embedded surface Σ in M is (the image of) an embedded J-holomorphic curve iff $J : T\Sigma \circlearrowleft$.

Definition 2. A J-holomorphic curve u is *multiply covered* if there exists

$$\begin{array}{ccc} (\Sigma, j) & \xrightarrow{\quad u \quad} & (M, \omega, J) \\ \big\downarrow p & \nearrow_{u'} & \\ (\Sigma', j') & & \end{array}$$

where the map p is a branched cover of degree > 1.

Recall a branched cover is locally modelled on $\mathbb{C} \to \mathbb{C}$ where $z \mapsto z^m$.

Definition 3. u is simple if it is not a multiple cover.

Note simple does not imply embedded, i.e. there can be singularities.

Definition 4. A J-holomorphic curve with k marked points consists of a holomorphic map

$$u : (\Sigma, j) \to (M, \omega, J)$$

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together with $z_1, \cdots, z_k \in \Sigma$ distinct modulo

$$\begin{array}{ccc} (\Sigma, j, z_1, \cdots, z_k) & \xrightarrow{u} & (M, \omega, j) \\ \downarrow & \nearrow_{u'} & \\ (\Sigma', j', z_1', \cdots, z_k') & & \end{array}$$

where the unlabelled map (which brings $z_i \mapsto z_i'$) is biholomorphic.

1.2. Examples.

Example 1. Algebraic curves in $\mathbb{CP}^2$ equipped with J_{std} are images of holomorphic maps.

Lemma 1. *If $u : (\Sigma, j) \to (M, \omega, J)$, Σ is connected, compact, and $u_* [\Sigma] = 0 \in H_2 (M; \mathbb{R})$ then u is constant.*

Proof. By the lemma from last lecture, $\operatorname{area}(u) = 0$. Choose local coordinate $z = s + it$ on Σ. This means

$$j \frac{\partial}{\partial s} = \frac{\partial}{\partial t} .$$

Locally the condition of being a holomorphic curve then means that

$$\frac{\partial u}{\partial t} = J \frac{\partial u}{\partial s}$$

and the area integrand can be written:

$$\begin{aligned} \text{integrand} &= \sqrt{g\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}\right) g\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) - g\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right)^2} \, |\, ds\ dt \,| \\ &= \sqrt{\omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}\right) \omega\left(\frac{\partial u}{\partial t}, J\frac{\partial u}{\partial t}\right) - \omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial t}\right)^2} \, |\, ds\ dt \,| \\ &= \sqrt{\omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right)^2} \, |\, ds\ dt \,| \\ &= \left|\frac{\partial u}{\partial s}\right|^2 |\, ds\ dt \,| = \left|\frac{\partial u}{\partial t}\right| |\, ds\ dt \,| \end{aligned}$$

but these are non-negative everywhere, so since the area is 0 we have

$$\frac{\partial u}{\partial s} \equiv \frac{\partial u}{\partial t} \equiv 0$$

which implies u is constant. □

Example 2. Fix a compact surface (Σ, j) and fix (M, ω_M, J_M). Consider the product

$$X = \left(\Sigma \times M, \underbrace{\omega_\Sigma \oplus \omega_M}_{\omega}, \underbrace{j \oplus J_M}_{J}\right) .$$

If $p \in M$ then we can look at maps

$$u_p : \Sigma \to X$$

which send $z \mapsto (z, p)$. This map is J-holomorphic.

Claim 1. Every J-holomorphic curve in X in the homology class $[\Sigma] \times [\{\mathrm{pt}\}]$ is of the form u_p.

Proof. Let $u' : (\Sigma', j') \to X$ be a J-holomorphic curve. Suppose $u'_* [\Sigma'] = [\Sigma'] \times [\mathrm{pt}]$. Then we want to show that u' is equivalent to some u_p.

Let $\pi : X \to M$ denote the projection. Then $\pi \circ u' : \Sigma' \to M$ is J_M-holomorphic and

$$(\pi \circ u')_* [\Sigma'] = 0 \ .$$

By the previous lemma, $\pi \circ u'$ is constant. Denote its image by $\{p\}$. Then

$$\mathrm{im}\,(u') \subseteq \Sigma \times \{p\} \ .$$

Let $\rho : X \to \Sigma$ denote the other projection. Then $\rho \circ u' : \Sigma' \to \Sigma$ is j-holomorphic. We have

$$(\rho \circ u')_* [\Sigma'] = [\Sigma] \ .$$

Then $\rho \circ u'$ is a degree 1 branch cover, and hence a diffeomorphism. So $\rho \circ u'$ is a biholomorphic map showing $u' \sim u_p$. So the diagram is

$$\begin{array}{ccc} \Sigma' & \xrightarrow{u'} & X \\ {\scriptstyle \rho \circ u'}\downarrow & \nearrow_{u_p} & \\ \Sigma & & \end{array} \ .$$

□

2. Gromov non-squeezing

2.1. **Strategy.** Start with a symplectic embedding $\varphi : B(r) \overset{s}{\hookrightarrow} Z(r)$ where

$$B(r) = \left\{ z \in \mathbb{C}^n \mid \pi |z|^2 \leq r \right\}$$

$$Z(r) = \left\{ z \in \mathbb{C}^n \mid \pi |z_1|^2 \leq R \right\} \ .$$

Now we go on a search for a holomorphic curve.

First we fix some notation. For $A \in H_2(X)$, we write

$$m^J_{g,k}(X, \omega, A)$$

for the collection of J-holomorphic curves u which have k marked points, are in the homology class A, and with domain a genus g compact Riemann surface. For $i = 1, \cdots, k$ we write ev_i for the evaluation map:

$$\begin{array}{ccc} m^J_{g,k} & \xrightarrow{\quad \mathrm{ev}_i \quad} & X \\ & & \\ (u, \Sigma, z_1, \cdots, z_k) & \longmapsto & u(z_i) \ . \end{array}$$

We can write $Z(R)$ as the product of the disk of area R with $\mathbb{C}^{n-1}$, and then we have a symplectic inclusion

$$Z(R) \hookrightarrow S^2(R + \epsilon) \times \mathbb{C}^{n-1}$$

where $S^2(R + \epsilon)$ is a 2-sphere with a symplectic form of area $R+\epsilon$. Let $\hat{\varphi} : B(r) \hookrightarrow S^2(R + \epsilon) \times \mathbb{C}^{n-1}$ be the composition. Since $B(r)$ is compact, its image is bounded. Let $T = \mathbb{C}^{n-1}/c\mathbb{Z}^{2n-2}$ where $c \gg 0$. Then the composition

$$\tilde{\varphi} : B(r) \hookrightarrow S^2(R + \epsilon) \times \mathbb{C}^{n-1} \to S^2 \times T$$

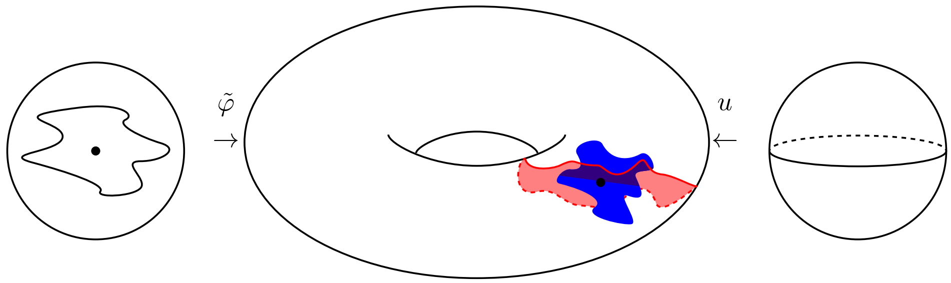

FIGURE 1. On the left we have $\tilde{\varphi} : B(r) \rightarrow S^2 \times T$ with blue image. Then we have a map from $u : S^2 \rightarrow S^2 \times T$. The image of this map (in red) contains the center of the image of the ball. Then the inverse image under u of the image of $\tilde{\varphi}$, Σ, is the domain of the map $\psi := \tilde{\varphi}^{-1} \circ u$. The region outlined inside of $B(r)$ is $\left(\tilde{\varphi}^{-1} \circ u\right)(\Sigma)$.

is still a symplectic embedding.

Let J be a compatible acs on $S^2(R+\epsilon) \times T$ such that on $\tilde{\varphi}(B(r))$ it agrees with $\tilde{\varphi}_*(J_0)$ where J_0 is the standard acs on $\mathbb{C}^n$. We can do this because the space of compatible acs is contractible.

Consider $m^J_{0,1}\left(S^2(R+\epsilon) \times T, \omega, \left[S^2\right] \times [\mathrm{pt}]\right)$. So these are holomorphic spheres in this manifold with one marked point. Then the evaluation map ev_1 maps

$$\mathrm{ev}_1 : m^J_{0,1}\left(S^2(R+\epsilon) \times T, \omega, \left[S^2\right] \times [\mathrm{pt}]\right) \rightarrow S^2(R+\epsilon) \times T \ .$$

The key point which we will prove next time is:

this map is surjective!

That is, for every $p \in S^2 \times T$, there is a J-holomorphic sphere in the class $\left[S^2\right] \times [\mathrm{pt}]$ "going through" p.

Gromov nonsqueezing follows (almost) immediately from this. We can deduce nonsqueezing as follows. Let $u : S^2 \rightarrow S^2 \times T$ be a curve given by the key fact with $\mathrm{im}(u) \ni \tilde{\varphi}(0)$, i.e. a curve containing the center of the ball. Let

$$\Sigma = u^{-1}(\tilde{\varphi}(B(r))) \subset S^2 \ .$$

Consider the map

$$\tilde{\varphi}^{-1} \circ u : \Sigma \rightarrow B(r) \ ,$$

which is actually J_0-holomorphic. The picture is as in fig. 1.

Now let's see what we ended up with.[1] The upshot is that, after some tweaking to get boundary transversality, we have (Σ, j) a compact Riemann surface with boundary, and a J_0-holomorphic map $\psi = \tilde{\varphi}^{-1} \circ u$ which maps

$$\Sigma \rightarrow B(r) \qquad \partial\Sigma \rightarrow \partial B(r) \ .$$

And in particular,

$$\mathrm{area}(\psi) < R + \epsilon \ .$$

Now we have the following lemma:

[1] Professor Hutchings says that at this point we should thank J-holomorphic curves and put some money in the tip jar.

Lemma 2 (Monotonicity lemma for minimal surfaces)**.** *Let Σ be a compact Riemann surface with boundary. If $\psi : \Sigma \to B(r)$, $0 \in \operatorname{im}(\psi)$, and ψ minimizes area relative to its boundary, then* $\operatorname{area}(\psi) \leq r$.

Such inequalities are sometimes called reverse isoperimetric inequalities.

Now by lemma 2, we have $\operatorname{area}(\psi) \leq r$, which implies $r < R + \epsilon$. But since ϵ is arbitrary we have $r \leq R$ so we are done.

So next time we will prove lemma 2, and then we have to think about why such a holomorphic curve exists, i.e. we will prove that this evaluation map is surjective.