LECTURE 15
MATH 242

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1. Moduli spaces of J-holomorphic curves

Recall last time we talked about

$$m^J_{g,k}(M,\omega,A)^*$$

which consists of the genus g J-holomorphic curves in M with k marked points and homology class A. The star indicates that they are simple, i.e. they are not multiply covered.

One of the claims made last time was

Claim 1. If J is generic then this is naturally a smooth manifold of dimension $(n-3)(2-2g)+2c_1(A)+2k$.

All of the details of this proof are in *J-holomorphic curves in symplectic topology* by McDuff-Salamon. We will prove the following simpler statement:

Claim 2. If J is generic, then $m^J_{(\Sigma,j)}(M,\omega,A)^*$ is naturally a smooth manifold of dimension $n(2-2g)+2c_1(A)$.

Remark 1. The dimension of the space of complex structures on Σ_g minus the dimension of $\mathrm{Aut}(\Sigma_g,j)$ is $6g-6$. For $g=0$ we have that the first term is 0, and the second is the dimension of the fractional linear transformations, which is 6. For $g=1$ both terms are 2. For $g>1$ the first term is $6g-6$ and the automorphism group is always finite so it has dimension 0.

$$\begin{array}{lrcr} & \dim(\text{complex structures on } \Sigma_g) & - & \dim(\mathrm{Aut}(\Sigma_g,j)) = 6g-6 \\ g=0: & 0 & - & 6=-6 \\ g=1: & 2 & - & 2=0 \\ g\geq 2: & 6g-6 & - & 0=6g-6 \end{array}$$

1.1. Being cut out transversely.

What does it mean for a holomorphic map $u:\Sigma\to M$ to be "cut out transversely"? Let's consider a finite dimensional model. Consider a smooth vector bundle

$$\begin{tikzcd} \mathbb{R}^k \arrow[r] & E \arrow[d, shift right] \\ & B^n \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

where ψ is a section. Then when is $\psi^{-1}(0)$ "cut out transversely"?

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Suppose $x \in \psi^{-1}(0)$. Then we have a well-defined map

$$D\psi : T_x B \to E_x$$

which is the "intrinsic derivative". We can check that these two possible definitions are the same:

Definition 1. Let ∇ be any connection and define $D\psi = \nabla\varphi$. If ∇' is a different connection then $(\nabla' - \nabla)\psi(x) = 0$. Note the tensor $(\nabla' - \nabla)$ is a bundle map $TB \to \operatorname{Hom}(E, E)$.

Definition 2. Define $D\psi$ to be the composition:

$$\begin{tikzcd} T_x B \arrow[r, "d\psi"] \arrow[rr, bend right, "D\psi"'] & T_{(x,0)}E = T_x B \oplus E_x \arrow[r, "\pi_2"] & E_x \end{tikzcd} .$$

We say that x is cut out transversely if $D\psi : T_x B \to E_x$ is surjective.

Proposition 1. *If $\psi^{-1}(0)$ is cut out transversely then $\psi^{-1}(0)$ is naturally a smooth manifold of dimension $n-k$. If $x \in \varphi^{-1}(0)$ then*

$$T_x \psi^{-1}(0) = \ker(D\psi : T_x B \to E_x) \ .$$

1.2. **Moduli spaces via vector bundles.** We can think of $m^J_{(\Sigma,j)}(M, \omega, A)$ as the zero set of the vector bundle $\mathcal{B}$ which consists of the smooth maps $u : \Sigma \to M$ representing the homology class A. This is an infinite dimensional Frechet manifold. For much of the analysis we will be doing we want to use separable Banach manifolds.

A Banach manifold is defined like a smooth manifold, except that coordinate charts map to open subsets of Banach spaces. For U and V open subsets of a Banach space, a map $f : U \to V$ is differentiable at $x \in U$ if there exists a bounded operator $D : E \to F$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - Dh}{\|h\|_E} = 0$$

where $D = df_x : E \to F$. Separable means that there exists a countable dense set.

Example 1. The space of C^k maps $\Sigma \to M$ in the class A is a Banach space. Maps in some Sobolev space are as well.

Remark 2. We will usually suppress this issue, but we mention it here so we are at least aware of it.

The bundle $\begin{matrix}\mathcal{E}\\ \downarrow \\ \mathcal{B}\end{matrix}$ is defined as follows. The base consists of maps $u : \Sigma \to M$ in the class A. The fibers look like

$$\mathcal{E}_u = \operatorname{Hom}^{0,1}(T\Sigma, u^*TM)$$

which consists of the bundle maps $\varphi : T\Sigma \to u^*TM$ such that $J\varphi + \varphi j = 0$ Now we define a section ψ by

$$\psi(u) = du - J \circ du \circ j \cdot T\Sigma \to u^*TM \ .$$

Explicitly we have

$$J\psi(u) + \psi(u) j = J\, du - du \circ j + du \circ j - J\, du \ .$$

Note that $\psi(u) = 0$ as a bundle map iff u is J-holomorphic. With this setup we have:

$$m^J_{(\Sigma,j)}(M,\omega,A) = \psi^{-1}(0) \ .$$

If u is a J-holomorphic map, then we have an intrinsic derivative

$$D\psi : T_u\mathcal{B} \to \mathcal{E}_u = \mathrm{Hom}^{0,1}(T\Sigma, u^*TM)$$

which is defined in the analogous way to the finite-dimensional case. As it turns out,

$$T_u\mathcal{B} = \Gamma(u^*TM) \ .$$

This $D\psi$ is just a linear map between spaces of sections, and is not necessarily a bundle map. In other words it is a first order differential operator. We denote this operator by D_u.

In local coordinates $z = s + it$ on Σ, and local coordinates on M, then

$$\begin{aligned} D_u(\eta) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \psi(u+\epsilon\eta) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (d(u+\epsilon\eta) + J\circ d(u+\epsilon\eta)\circ j) \\ &= d\eta + (\eta J)\circ du \circ j + J \circ d\eta \circ j \end{aligned}$$

where η is a map from the coordinate path on Σ to the coordinate patch on M. The idea is that the first and last terms are $\bar{\partial}\eta$, and the middle term is a 0th order error term since

$$\bar{\partial}\eta = \left(\frac{\partial u}{\partial s} + i\frac{\partial \eta}{\partial t}\right)(ds - i\,dt) \ .$$

The upshot is that in local coordinates $D_u\eta = \bar{\partial}\eta +$ some 0th order term.

1.3. **Differential operators.**

Definition 3. Suppose E and F are smooth vector bundles over a smooth finite dimensional manifold M. A *Differential operator* $D : \Gamma(E) \to \Gamma(F)$ is a linear map which in local coordinates and trivializations takes the form

$$D\psi = \sum_{|I|\le K} a_I \frac{\partial}{\partial x^I}\psi$$

where $I = (i_1, \cdots, i_m)$,

$$\frac{\partial}{\partial x^I} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_m}} \ ,$$

and for any open set $U \subseteq M$

$$E|_U \simeq U \times \mathbb{R}^p \qquad\qquad F|_U \simeq U \times \mathbb{R}^q$$

and

$$a_I : U \to \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$$

is a smooth function.

Then the statement is that D_u is a first order differential operator

$$\Gamma({}^*TM) \to \Gamma\left(\mathrm{Hom}^{0,1} * (T\Sigma, u^*TM)\right) \ .$$

The kth order part of a differential operator of order k is a well-defined bundle map

$$\mathrm{Sym}^k(T^*M) \to \mathrm{Hom}(E,F) \ .$$

This bundle map is called the *symbol* of D, denoted by $\sigma(D)$.

Example 2. Locally we have seen that

$$D_u\eta = \left(\frac{\partial \eta}{\partial s} + i\frac{\partial \eta}{\partial t}\right)(\, ds \, - i \, dt \,) + (\text{0th order term}) \ .$$

The symbol is

$$\begin{aligned} \sigma(D_u) : T\Sigma &\longrightarrow \operatorname{Hom}\left(u^*TM, \operatorname{Hom}^{0,1}(\Sigma, u^*TM)\right) \\ ds &\longmapsto (\, ds \, - i \, dt \,) \otimes \mathrm{id}_{u^*TM} \\ dt &\longmapsto i\,(\, ds \, - i \, dt \,) \otimes \mathrm{id}_{u^{TM}} \end{aligned} \ .$$

The following type of operator is somehow nice:

Definition 4. A differential operator is *elliptic* if whenever $\xi \in T^*M$ is nonzero, we have that

$$\sigma(D)\underbrace{(\xi, \cdots, \xi)}_{k}$$

is an isomorphism.

Then we have the following theorem:

Theorem 1. *If D is elliptic, then D is Fredholm (on suitable Banach space completions) which means that D has closed range and* $\dim\ker D, \dim\operatorname{coker} D < \infty$. *In addition, every element of* $\ker D$ *in the Banach space completion is actually smooth. This is called elliptic regularity.*

There is a nice formula for the index:

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D$$

which comes from the Atiyah-Singer index theorem. We won't even state this right now[1] but we will see what it gives us in this scenario.

Definition 5. Let E be a rank 1 complex vector bundle over a genus g Riemann surface (Σ, j). A *Cauchy-Riemann (CR) operator* on E is a first-order differential operator

$$D : \Gamma(E) \to \Gamma\left(\operatorname{Hom}^{0,1}(T\Sigma, E)\right)$$

such that in local coordinates

$$D = \left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right)(\, ds \, - i \, dt \,) + (\text{0th order term}) \ .$$

For example, D_u is a CR operator with $E = u^*TM$. Note that any CR operator is elliptic.

Theorem 2 (Generalized Riemann-Roch formula)**.**

$$\operatorname{ind}(D) = n \underbrace{\chi(\Sigma)}_{(2-2g)} + 2c_1(E)$$

.

Example 3. $\operatorname{ind}(D_u) = n(2-2g) + 2c_1(A)$.

[1] Because of the inevitable digression into K-theory.

So this all comes together to tell us that u is cut out transversely if D_u is surjective. In this case, the implicit function theorem tells us that the moduli space is a manifold near u of dimension $\operatorname{ind}(D_u)$ and $T_u M = \ker D_u$.

Next time we will show that if J is generic, then D_u is surjective for all $u \in m^*$.