LECTURE 16
MATH 242

LECTURE: PROFESSOR MICHAEL HUTCHINGS
NOTES: JACKSON VAN DYKE

1. Transversality

Recall we were discussing transversality of holomorphic curves. Fix (Σ, j) some compact Riemann surface of genus g. Recall a holomorphic curve is a map

$$u : (\Sigma, j) \to (M, J)$$

such that

$$J \circ du = du \circ j : T_p\Sigma \to T_{u(p)}M \ .$$

We can rewrite this to be:

$$du + J \circ du \circ j = 0$$

and then the derivative of this equation is a first order differential operator

$$D_u : \Gamma(u^*TM) \to \Gamma\left(T^{0,1}\Sigma \otimes_{\mathbb{C}} u^*TM\right) = \mathrm{Hom}^{0,1}(T\Sigma, u^*TM) \ .$$

In a local coordinate $z = s + it$ on Σ and local trivialization of u^*TM we have

$$D_u\psi = (\partial_s\psi + i\partial_t\psi + A(s,t)\psi) \otimes (ds - i\,dt)$$

where A is a $2n \times 2n$ real matrix.

Recall that D_u is Fredholm and

$$\mathrm{ind}(D_u) = n(2 - 2g) + 2c_1(u^*TM) \ .$$

If D_u is surjective, then the moduli space of holomorphic maps

$$m^J_{(\Sigma,j)}(M, \omega, J)$$

is naturally a manifold near u of dimension $\mathrm{ind}(D_u)$ with $T_u m = \ker(D_u)$. In this case we say that u is "cut out transversely" (as a holomorphic map).

Example 1. Let $n > 1$. Suppose $M = \Sigma \times N^{2n-2}$ with $\omega_M = \omega_\Sigma \oplus \omega_M$ and $J = j \oplus J_N$. Take

$$u : \Sigma \to M$$

to take $z \mapsto (z, p)$ where $p \in N$ is fixed. Then we have

$$D_u : \Gamma(u^*TM) \to \Gamma\left(T^{0,1}\Sigma \otimes u^*TM\right)$$

where

$$u^*TM = T\Sigma \oplus \underline{T_pN} = T\Sigma \oplus \underline{\mathbb{C}^{n-1}}$$

where the second term is the trivial bundle with fiber $\mathbb{C}^{n-1}$. Note that D_u respects the direct sum decomposition and is $\bar\partial$ on each factor.

Now the question is if D_u is surjective. First of all the index is

$$\mathrm{ind}(D_u) = n(2 - 2g) + 2(2 - 2g) = (n + 2)(2 - 2g) \ .$$

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If $g > 0$, then $\operatorname{ind}(D_u) \leq 0$. We also have that $\dim\ker D_u \geq 2n-2$ which implies $\dim\operatorname{coker} D_u > 0$ which implies D_u is not surjective. In fact if we perturb J then the holomorphic curves disappear except maybe finitely many when $g = 0$.

Claim 1. If $g = 0$ then D_u is surjective.

Proof. In this case, $\operatorname{ind}(D_u) = (2n-2) + 6$ and

$$\dim\ker(D_u) = (2n-2) + 6$$

where the first term comes from constant sections of T_pN and the second term comes from the space of infinitesimal automorphisms of S^2 (flt). This means there cannot be any cokernel. □

1.1. Useful facts about Cauchy Riemann type operators.

These facts are actually true for Cauchy-Riemann (CR) type operators, i.e. those which look like $\bar{\partial}$ plus a zero order term, but we will primarily be interested in the D_u case.

Fact 1 (Unique continuation)**.** *Let D be a CR operator. If $D\psi \equiv 0$ and ψ vanishes to infinite order at a point, then $\psi \equiv 0$.*

Fact 2. *If $D : \Gamma(L) \to \Gamma\left(T^{0,1}\Sigma \otimes L\right)$ is a CR operator where L is a line bundle, if $D\psi \equiv 0$ then every zero of ψ is isolated and has positive multiplicity.*

In particular, if $\ker D \neq 0$ then $c_1(L) \geq 0$ with equality iff this section is non-vanishing.

Both facts are proved using the "Carleman similarity principle." Roughly speaking, this says that near a zero of ψ, there is a trivialization in which ψ is a holomorphic function.

Remark 1. Let $E, F \to M$ be two bundles (equipped with metrics) over a compact Riemannian manifold M. Consider some differential operator

$$D : \Gamma(E) \to \Gamma(F) \ .$$

Then we have a formal adjoint

$$D^* : \Gamma(F) \to \Gamma(E)$$

such that

$$\int_M \langle D\psi, \eta\rangle = \int_M \langle \psi, D^*\eta\rangle \ .$$

For the CR operator, we have that locally

$$D \sim_{\text{loc}} \partial_s + i\partial_t + A(s,t) \qquad\qquad D^* \sim_{\text{loc}} -\partial_s + i\partial_t + A(s,t)^* \ .$$

If $D^*\eta = 0$ and $\eta \not\equiv 0$, then by fact 2, any zero of η is isolated and has negative multiplicity. So if a CR operator

$$D : \Gamma(L) \to \Gamma\left(T^{0,1}\Sigma \otimes L\right)$$

has a nonzero cokernel, then

$$c_1\left(T^{0,1}\Sigma \otimes L\right) = \chi(\Sigma) + c_1(L) = 2 - 2g + c_1(L) \leq 0 \ .$$

Therefore if $2 - 2g + c_1(L) > 0$ then D is automatically surjective.

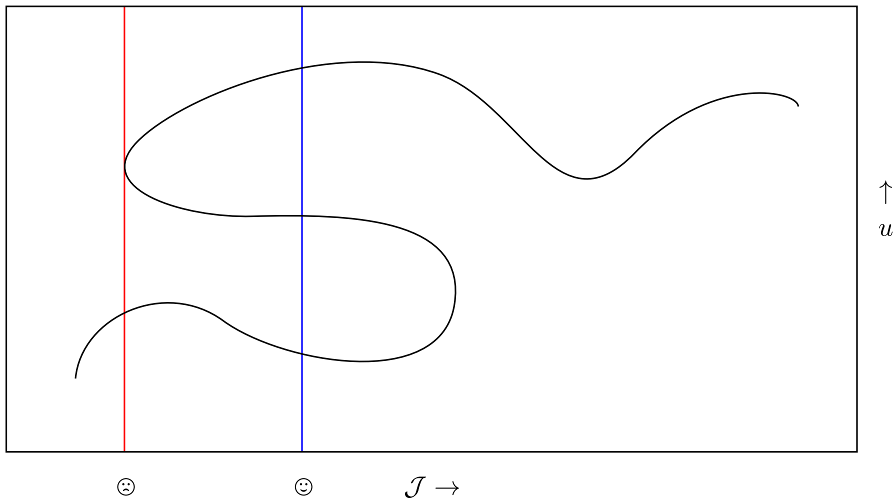

FIGURE 1. The horizontal axis is the space $\mathcal{J}$, and the vertical axis consists of all maps $u : \Sigma \to M$. Then the idea is that the blue fiber is some m^J which is cut out transversely, and the red fiber is not.

Example 2. Consider the product curves in $S^2 \times N$. In this case we had

$$D_u : \Gamma\left(TS^2 \oplus \underline{T_pN}\right) \to \Gamma\left(T^{0,1}S^2 \otimes (\cdots)\right) .$$

Then this splits as

$$D_u : \Gamma\left(TS^2\right) \to \Gamma\left(T^{0,1}S^2 \otimes S^2\right)$$

plus $n-1$ copies of

$$D_u : \Gamma\left(\underline{\mathbb{C}}\right) \to \Gamma\left(T^{0,1}S^2 \otimes \underline{\mathbb{C}}\right) .$$

We argued directly that the operator was surjective. But now we can see that in the first term we have $2 - 2g + c_1(L) = 4$ and in the second $2 - 2g + c_1(L) = 2$ so by the above remark this is surjective.

2. Transversality theorem

Theorem 1. *If J is generic, them $m^J_{(\Sigma,j)}(M,\omega)^*$ is cut out transversely.*

Generic means that the set of J with this property is a countable intersection with open dense sets in the space of all J. By the Baire category theorem this implies the set of such J is dense. The full details are in big McDuff Salamon chapter 3.

Outline of proof. Let $\mathcal{J}$ be the set of all ω-compatible acs. Define the universal moduli space

$$\mathcal{U} = \left\{(J,u) \mid J \in \mathcal{J}, u \in m^J_{(\Sigma,j)}(X,\omega)^*\right\} .$$

The first step is to prove that $\mathcal{U}$ is cut out transversely. There is a projection $\pi : \mathcal{U} \to \mathcal{U}$ which sends $(J,u) \mapsto J$. The Sard-Smale theorem tells us that a generic J is a regular value of π. The picture is as in fig. 1.

Now we show how to prove $\mathcal{U}$ is cut out transversely. Consider

$$\begin{array}{c}\mathcal{E}\\ {\scriptstyle\psi}\circlearrowleft\big\downarrow\\ \mathcal{J}\times\operatorname{Maps}(\Sigma, M)^*\end{array}$$

where

$$\mathcal{E}(J,U) = \Gamma\left(T^{0,1}\Sigma\otimes u^*TM\right) \qquad \psi(J,u) = du + J\circ du\circ j\ .$$

By definition $\mathcal{U} = \psi^{-1}(0)$. By the implicit function theorem, in order to show that $\mathcal{U}$ is cut out transversely we need to show that if $(J,u)\in\psi^{-1}(0)$ then

$$D\psi : T_{(J,u)}(\mathcal{J}\times\text{Maps}) \to \mathcal{E}_{(J,u)}$$

is surjective. We have that

$$D\psi : T_J\mathcal{J}\oplus\Gamma(u^*TM)\to\Gamma\left(T^{0,1}\Sigma\otimes u^*TM\right)\ .$$

The idea is somehow that we haven't changed the codomain, we have just enlarged the domain to make it easier to prove surjectivity. Explicitly we have that

$$\boxed{D\psi\left(\dot{J},\eta\right) = \dot{J}\circ du\circ j + D_u\eta}\ .$$

The cokernel of this is at most finite dimensional and the range is closed. So to prove this is surjective it is enough to show that if ζ is perpendicular to the image of $D\psi$, then $\zeta = 0$. So suppose $\zeta\perp\operatorname{im}(D\psi)$. In particular this means $\zeta\perp\operatorname{im}(D_u)$, so $\zeta\in\ker(D_i^*)$. By Carleman similarity (unique continuation) it is enough to show that ζ vanishes to infinite order at some point. So now we need some lemmas.[1]

Lemma 1. *The fact that u is simple means that u is somewhere injective, i.e. there is $z\in\Sigma$ such that du_z is injective and $u^{-1}(u(z)) = z$.*

Lemma 2. *There exists some $\dot{J}$ which is supported in an arbitrarily small neighborhood of z such that $\left(\dot{J}\circ du\circ j\right)(z) = \zeta(z)$. In other words we have the freedom to move J such that as long as du is injective $\dot{J}\circ du\circ j$ can be anything we want.*

Then $\zeta\perp D_u\left(\dot{J},0\right)$ implies that $\zeta(z) = 0$.

This implies that ζ vanishes on a nonempty open set (the somewhere injective points) so $\zeta\equiv 0$. $\square$

[1] See McDuff-Salamon for the proofs.