LECTURE 18
MATH 242

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Today we will continue learning about the nice things Gromov did.

1. Recognizing $\mathbb{R}^4$

Theorem 1. *Let $\left(X^4, \omega\right)$ be a symplectic 4-manifold with $H_*(X) = H_*(pt)$. Suppose there exist compact sets $K \subseteq X$ and $L \subseteq \mathbb{R}^4$ and a symplectomorphism*

$$\varphi : (X \setminus K, \omega) \xrightarrow{\simeq} \left(\mathbb{R}^4 \setminus L, \omega_{std}\right) \ . \tag{1}$$

Then there exists a compact set $K \subseteq K' \subseteq X$ and a symplectomorphism

$$\psi : (X, \omega) \xrightarrow{\simeq} \left(\mathbb{R}^4, \omega_{std}\right) \tag{2}$$

such that $\psi|_{X \setminus X'} = \varphi$.

This is useful for the following. Recall we have the following conjecturE:

Conjecture 1 (Smooth 4d Poincaré)**.** *If M is a compact smooth 4-manifold, homotopy equivalent to S^4, then M is diffeomorphic to S^4.*

Then by theorem 1 one would just[1] need to show that $M \setminus \{\text{pt}\}$ has a symplectic form which is standard near the point.

2. Special properties of holomorphic curves in four-dimensions

2.1. Intersection positivity.

Intersection positivity is the following. Given $\left(X^4, \omega\right)$ with J compatible, let

$$u_1 : (\Sigma_1, j_1) \to X \qquad\qquad u_2 : (\Sigma_2, j_2) \to X \tag{3}$$

be J-holomorphic curves such that Σ_1, Σ_2 are connected, and u_1, u_2 are somewhere injective ($\iff$ simple), and $u_1(\Sigma_1) \neq u_2(\Sigma_2)$. Then:

- intersections of $u(\Sigma_1)$ and $u(\Sigma_2)$ are isolated. In fact the set of $(z_1, z_2) \in \Sigma_1 \times \Sigma_2$ such that $u_1(z_1) = u_2(z_2)$ is isolated in $\Sigma_1 \times \Sigma_2$.
- Each intersection point has positive multiplicity, with multiplicity 1 iff the intersection is transverse.

Proof of trivial case. The following is the statement of the trivial case:

Claim 1 (Trivial case)**.** Let u_i be immersions with $u_1(z_1) = u_2(z_2)$ a transverse intersection. Let $\{v_1, w_1\}$ be an oriented basis for $T_{z_11}\Sigma_1$, and $\{v_2, w_1\}$ be an oriented basis for $T_{z_2}\Sigma_2$. Then the intersection has multiplicity $+1$ iff

$$\{\, du_1\ (v_1)\,, du_1\ (w_1)\,, du_2\ (v_1)\,, du_2\ (v_2)\} \tag{4}$$

[1] This would win you the Fields medal, and the Breakthrough prize. Professor Hutchings says he isn't trying for this anymore because the year Cameron Diaz handed it out already passed.

is an oriented basis for T_pX.

This holds because $\mathrm{Span}\left(du_1\left(v_1\right), du_1\left(w_1\right)\right)$ and $\mathrm{Span}\left(du_2\left(v_2\right), du_2\left(w_2\right)\right)$ are $\mathbb{C}$-linear subspaces of T_pX and $\mathrm{GL}_2\left(\mathbb{C}\right)$ is connected. E.g. if Σ_1, Σ_2 are closed and

$$u_{1*}\left[\Sigma_1\right] \cdot u_{2*}\left[\Sigma 2\right] = 0 \tag{5}$$

then $u_1\left(\Sigma_1\right) \cap u_2\left(\Sigma_2\right) \neq \emptyset$. □

2.2. **Adjunction formula.** Let

$$u : \left(\Sigma, j\right) \to \left(X^4, \omega, J\right) \tag{6}$$

be holomorphic and simple, where Σ is closed and connected. Write $A = u_*\left[\Sigma\right] \in H_2\left(X\right)$. Then

- the singularities[2] of $u\left(\Sigma\right)$ are isolated, and
- we have the formula:

$$\left\langle c_1\left(TX\right), A\right\rangle = \chi\left(\Sigma\right) + A \cdot A - 2\delta\left(u\right) \tag{7}$$

where $\delta\left(u\right)$ is a count of the singularities of $u\left(\Sigma\right)$ with positive integer multiplicities.

Proof in special case. Let u be an immersion and assume the only singularities are transverse double points. Since u is an immersion, we have

$$u^*TX = T\Sigma \oplus N \tag{8}$$

as complex vector bundles where N is the normal bundle. Now we have the following property of c_1:

$$c_1\left(u^*TX\right) = c_1\left(T\Sigma\right) + c_1\left(N\right) \in H^2\left(\Sigma; \mathbb{Z}\right) \ . \tag{9}$$

Now we can evaluate both sides on $\left[\Sigma\right] \in H_2\left(\Sigma\right)$ to get

$$\left\langle c_1\left(u^*TX\right), \left[\Sigma\right]\right\rangle = \chi\left(\Sigma\right) + \left\langle \Sigma\right\rangle + \left\langle c_1\left(N\right), \left[\Sigma\right]\right\rangle \ . \tag{10}$$

So we need to show that

$$\left\langle c_1\left(N\right), \left[\Sigma\right]\right\rangle = A \cdot A - 2\delta\left(u\right) \ . \tag{11}$$

Let ψ be generic section of N, and let $u' : \Sigma \to X$ be the map sending $z \mapsto \exp_{u(z)} \psi\left(z\right)$. Then

$$\left\langle c_1\left(N\right), \left[\Sigma\right]\right\rangle = \#\psi^{-1}\left(0\right) = \#\left(u \cap u'\right) = A \cdot A \ . \tag{12}$$

But we over-count when $u\left(\Sigma\right)$ intersects itself, so we actually have

$$\left\langle c_1\left(N\right), \left[\Sigma\right]\right\rangle = \#\psi^{-1}\left(0\right) = \#\left(u \cap u'\right) - 2\delta\left(u\right) = A \cdot A - 2\delta\left(u\right) \tag{13}$$

and we are done. □

Now the following key proposition puts this to use:

Proposition 1. *Consider*

$$\left(X, \omega\right) = \left(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2}\right) \tag{14}$$

where these forms have the same area on S^2.[3] *Let J be any ω-compatible acs. Then $S^2 \times S^2$ has a foliation by embedded J-holomorphic spheres in the class $\left(1, 0\right) \in$*

[2]point where u is not locally an embedding.

[3]This is critical.

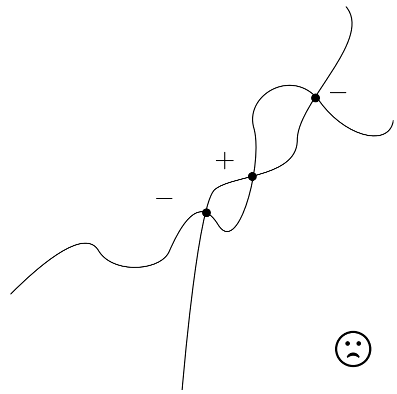

FIGURE 1. This is exactly what we have to show cannot happen.

$H_2\left(S^2 \times S^2\right)$ (which we call A-sphere) and a foliation by embedded J-holomorphic spheres in the class $(0,1)$ (which we call B-spheres).

If $J = J_1 \oplus J_2$ then we already know this. The A-spheres are just $S^2 \times \mathrm{pt}$, and the B-sphere are just $\mathrm{pt} \times S^2$. The picture is somehow that in this almost trivial case, it looks like a cartesian grid, and in general these lines will be somehow warped without crossing as in fig. 1.

Proof. We will construct both foliations in the same way so we just do the A-sphere case. Consider the evaluation map

$$m^J_{0,1}\left(S^2 \times s^2, (1,0)\right) \xrightarrow{\text{ev}} S^2 \times S^2 \ . \tag{15}$$

What do we know about holomorphic spheres in the class $(1,0)$?

- They are simple, because the homology class $(1,0)$ is indivisible.
- From the adjunction formula, since $u : S^2 \times S^2 \times S^2$ is in class $A = (1,0)$, we get

$$\underbrace{\left\langle c_1\left(T\left(S^2 \times S^2\right)\right), (1,0)\right\rangle}_{2} = \underbrace{\chi\left(S^2\right)}_{2} + \underbrace{(1,0)\cdot(1,0)}_{0} - 2\delta(u) \tag{16}$$

  which means $\delta(u) = 0$, so there are no singularities, which means u is an embedding.
- We get automatic transversality, i.e. $u : S^2 \to S^2 \times S^2$ in the class $(1,0)$ is always cut out transversely. This is because of the following. For an immersed holomorphic curve $u : \Sigma \to X^4$ and linearized operator

$$D_u : \Gamma(N) \to \Gamma\left(T^{0,1}\Sigma \otimes N\right) \tag{17}$$

  we have that u is cut out transversely when this is surjective. In our case, $D_u : \Gamma(N) \to \Gamma\left(T^{0,1}S^2 \otimes N\right)$, but N is trivial, and

$$\left\langle c_1\left(T^{0,1}S^2 \otimes N\right), \left[S^2\right]\right\rangle = 2 > 0 \tag{18}$$

  implies $\ker\left(D_u^+\right) = 0$ so the u are cut out transversely.

This all means $m^J_{0,1}\left(S^2 \times S^2, (0,1)\right)$ is a smooth manifold of dimension 4. But is it compact? The symplectic area of the class $(a,b) \in H_2\left(S^2 \times S^2\right)$ for $a, b \in \mathbb{Z}$ is a constant multiple of $a+b$. So the class $(1,0)$ has minimal positive symplectic area, so no bubbling is possible, so this space is compact.

Now we have

$$\deg\left(\mathrm{ev} : m_{0,1}^J\left(S^2\times S^2,(1,0)\right)\xrightarrow{\mathrm{ev}} S^2\times S^2\right)=1 \tag{19}$$

which implies ev is injective. This is because of the following. Suppose $p\in S^2\times S^2$ has more than one inverse image. Then there are two distinct holomorphic curves in the class $(1,0)$ intersection at p. By intersection positivity, $(1,0)\cdot(1,0)>0$ so we have a contradiction. Modulo some details we are done. □