LECTURE 19
MATH 242

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Recall last time we had the following:

Proposition 1. *Let J be any compatible acs on $\left(S^2 \times S^2, \omega \oplus \omega\right)$. Then $S^2 \times S^2$ has a foliation by J-holomorphic sphere in the class $(1,0) \in H_2\left(S^2 \times S^2\right)$ (A-spheres) and a foliation by J-holomorphic spheres in the class $(0,1) \in H_2\left(S^2 \times S^2\right)$ (B-sphere). Each A-sphere transversely intersects each B-sphere at a single point.*

We introduced this to prove recognition of $\mathbb{R}^4$, but we will first use it to show another theorem of Gromov:

Theorem 1 (Gromov)**.** *The inclusion*

$$\mathrm{SO}(3) \times \mathrm{SO}(3) \hookrightarrow \mathrm{Symp}_0\left(S^2 \times S^2, \omega \oplus \omega\right)$$

(where the symplectic forms ω have equal area, and Symp_0 denotes the identity component of the symplectomorphism group) is a homotopy equivalence.

Proof. Fix $p \in S^2$. Let X be the set of triples:

$$X = \{(J, \varphi_1, \varphi_2)\}$$

where J is a compatible acs $S^2 \times S^2$, and the φ_i are symplectomorphisms:

$$\begin{aligned} \varphi_1 :& S^2 \xrightarrow{\simeq} (A\text{-sphere through } p) \\ \varphi_2 :& S^2 \xrightarrow{\simeq} (B\text{-sphere through } p) \ . \end{aligned}$$

Note that X is homotopy equivalent to $\mathrm{SO}(3) \times \mathrm{SO}(3)$.

Now we define

$$f : X \to \mathrm{Symp}_0\left(S^2 \times S^2\right) \ .$$

Given $(J, \varphi_1, \varphi_2) \in X$ define $\psi \in \mathrm{Symp}_0\left(S^2 \times S^2\right)$ as follows. First define $\psi_0 \in \mathrm{Diff}_0\left(S^2 \times S^2\right)$ to be

$$\{\psi_0(z_1, z_2)\} = (B\text{-sphere through } \varphi_1(z_1)) \cap \cap (A\text{-sphere through } \varphi_2(z_2)) \cap$$

The picture is as in fig. 1.

So ψ_0 is a diffeomorphism, but it might not be symplectic, so we have to fix it. In particular we will fix this using the Moser trick.

Lemma 1. $(\omega \oplus \omega) \wedge \psi_0^*(\omega \oplus \omega) > 0$.

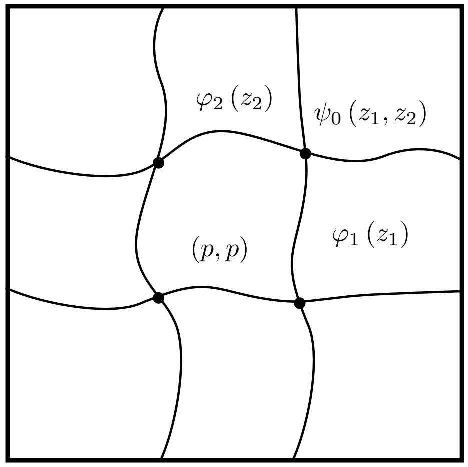

FIGURE 1. The picture of our diffeomorphism ψ_0.

Proof. Given $(q_1, q_2) \in S^2 \times S^2$, choose symplectic bases (v_1, w_1) for $T_{q_1} S^2$, and (v_2, w_2) for $T_{q_2} S^2$.

$$\begin{aligned}[(\omega \oplus \omega) \wedge \psi_0^* (\omega \oplus \omega)] (v_1, w_1, v_2, w_2) &= \psi_0^* (\omega \oplus \omega) (v_1, w_1) + \psi_0^* (\omega \oplus \omega) (v_2, w_2) \\ &= (\omega \oplus \omega) ((\psi_1)_* v_1, (\psi_0)_0 w_1) \\ &+ (\omega \oplus \omega) ((\psi_1)_* v_2, (\psi_0)_0 w_2) > 0\end{aligned}$$

□

Now define

$$\omega_t = t (\omega \oplus \omega) + (1 - t) \psi_0^* (\omega \oplus \omega) \ .$$

Then

$$\omega_t \wedge \omega_t = t^2 + 2t (1 - t) + (1 - t)^2 > 0$$

which implies ω_t is symplectic for $t \in [0, 1]$.

By the Moser trick there is a canonical isotopy $\{\psi_t\}_{t \in [0,1]}$ from ψ_0 to ψ_1 with

$$\psi_1^* (\omega \oplus \omega) = \omega \oplus \omega \ .$$

Define $\psi = \psi_1$.

Now define

$$g : \mathrm{Symp}_0 \left(S^2 \times S^2\right) \to X$$

by

$$g (\psi) = (\psi_* (J_0 \oplus J_0), \varphi_1, \varphi_2 \text{ given by } \psi) \ .$$

Now if we look at the construction we have $f \circ g = \mathrm{id}_{\mathrm{Symp}_0}$, and then we have a homotopy equivalence $g \circ f \sim_{\mathrm{hom}} \mathrm{id}_X$. ■

Remark 1. The theorem is false if the two symplectic forms have different areas.

Example 1. Consider $\left(S^2 \times S^2, \omega_1 \oplus \omega_2\right)$, where ω_1 has area 1 and ω_2 has area $1 + \epsilon$. In this case Gromov compactness can fail when constructing the A and B sphere.

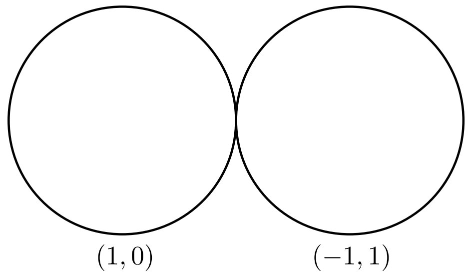

Figure 2. Bubbling in the case when the two spheres do not have equal areas.

For example, in $H_2\left(S^2 \times S^2\right)$, we have

$$(0,1) = (1,0) + (-1,1) \ .$$

The LHS has area $1+\epsilon$, the first term on the RHS has area 1, and the second term on the RHS has area ϵ, so we could have bubbling as in fig. 2.

Sometimes bubbling can be ruled out with an index argument. For example when we have $A = (-1,1)$,

$$\text{ind} = \underbrace{(n-3)\,\chi}_{=-2} + 2\underbrace{c_1\left(A\right)}_{=0} \ .$$

We can actually quantify this. Consider

$$\Sigma = \left\{J \in \mathcal{J} \mid \exists J-\text{holomorphic curve in the class } (-1,1)\right\} \ .$$

This is a codimension 2 “subvariety” of $\mathcal{J}$. Map

$$\Phi : \pi_1\left(\text{Symp}_0\left(S^2 \times S^2, \omega_1 \oplus \omega_2\right)\right) \to \mathbb{Z}$$

by taking $\Phi\left(\psi_t\right)$ to be the linking number of $\left(\psi_t\right)_*\left(J_0 \oplus J_0\right)$ with Σ. It is known that this map is nontrivial in explicit examples. So we get a different answer in the case of different areas.

This is extensively studied by McDuff, Abreu, and others.

1. Recognition of $\mathbb{R}^4$

Recall the theorem said the following. Let $\left(X^4, \omega\right)$ be a noncompact symplectic 4-manifold with $K \subset X$ compact and $H_*\left(X\right) \simeq H_*\left(\text{pt}\right)$. Let $L \subset \mathbb{R}^4$ compact. Then the existence of

$$\varphi : \left(X \setminus K, \omega\right) \xrightarrow{\simeq} \left(\mathbb{R}^4 \setminus L, \omega_{\text{std}}\right)$$

implies there exists a symplectomorphism

$$\left(X, \omega\right) \xrightarrow{\simeq} \left(\mathbb{R}^4, \omega_{\text{std}}\right)$$

which agrees with φ outside a compact set $K' \supset K$.

Proof. By enlarging K, we can assume WLOG that

$$L = D^2 \times D^2$$

where these D^2s have the same area since L is the complement of the image of the complement of K. The point is that inside K anything could be happening, but L is just a sort of rectangle. We can complete (L, ω_{std}) to $\left(S^2 \times S^2, \omega_1 \oplus \omega_2\right)$ where the ω_i have the same area by adding a bit, which we call Θ. Likewise we can complete (K, ω) to $\left(\bar{K}, \bar{\omega}\right)$. Since X is homologically trivial,

$$H_* \left(\bar{K}\right) \simeq H_* \left(S^2 \times S^2\right)$$

and

$$[\bar{\omega}] = (a, a) \in H^2 \left(\bar{K}; \mathbb{R}\right) \ .$$

Fix an $\bar{\omega}$-compatible acs $\bar{J}$ on $\bar{K}$. A generalization of the proposition from before shows that $\bar{K}$ has foliations by $\bar{K}$-holomorphic spheres in the class $(1, 0)$ (A-sphere) and in the class $(0, 1)$ (B-spheres). To prove this, assume $\bar{J}$ is a product acs on Θ. As in the proof of the proposition

$$m_{0,1}^{\bar{J}} \left(\bar{J}, (1, 0)\right) \qquad m_{0,1}^{\bar{J}} \left(\bar{J}, (0, 1)\right)$$

are compact and consist of embedded holomorphic spheres which give a foliation of some subset of $\bar{K}$. Since we have such spheres in Θ, the evaluation maps from these moduli spaces to $\bar{K}$ have degree 1, and we get foliations of all of $\bar{K}$.

As before, we get a symplectomorphism $\bar{K} \to S^2 \times S^2$ by using the A-spheres and B-spheres to define a diffeomorphism using the Moser trick. With more work, we can get this symplectomorphism to be the identity on Θ. Can remove boundaries and extend to get a symplectomorphism $X \simeq \mathbb{R}^4$.

The idea is that these foliations sort of look like coordinates. $\square$