LECTURE 21
MATH 242

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1. Morse theory

1.1. **Topology on the compactified moduli space.** The compactification of a moduli space of flow lines is defined to be

$$\overline{m(p,q)} = \coprod_{\substack{k\geq 1 \\ p=p_0\neq p_1\neq \ldots \neq p_k = q}} m(p_0,p_1)\times \ldots \times m(p_{k-1},p_k) \ .$$

Let $\{\gamma_k\}_{k=1}$ be a sequence in $\tilde{m}(p,q)$. We say that the sequence $\{[\gamma_i]\}$ in $m(p,q)$ converges to

$$([\eta_1],\ldots,[\eta_k]) \in m(p_0,p_1)\times \ldots \times m(p_{k-1},p_k)$$

if there are $s_{i,j}\in\mathbb{R}$ with $s_{i,1} > s_{i,2} > \ldots > s_{i,k}$ such that

$$\gamma_i(s_{i,j}+\cdot) \to \eta_j$$

in $\mathcal{C}^\infty$ on compact sets.

Exercise 1. Let X be a compact manifold. Any sequence in $m(p,q)$ has a subsequence which converges in $\overline{m(p,q)}$.

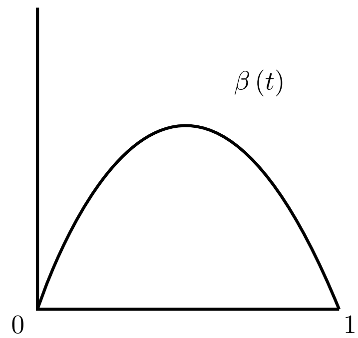
1.2. **Invariance.** So we know the Morse homology is equivalent to the usual homology, and we sketched why this is true. But given X, how can we prove that the Morse homology $H_*(X,f,g)$ does not depend on the Morse-Smale pair (f,g)? For example, if some aliens came up with a parallel version of mathematics and invented Morse homology before algebraic topology, how might they prove that this is invariant?[1] Let (f_0,g_0) and (f_1,g_1) be two Morse-Smale pairs. The space of smooth functions $f: X\to\mathbb{R}$ and metrics g on X are both contractible. Let $\{(f_t,g_t)\}_{t\in[0,1]}$ be a smooth path from (f_0,g_0) to (f_1,g_1).

Remark 1. We cannot expect that (f_t,g_t) is Morse-Smale for all t.

We do want the path $\{(f_t,g_t)\}$ to be suitable generic.

Approach 1: bifurcation analysis. There are only finitely many t for which (f_t,g_t) is not Morse-Smale. So we can study how Morse complex changes and check that the homology stays the same. We can do this, and it's sort of interesting, but there is another approach which is better for some purposes.

[1] Professor Hutchings says one can find alien mathematics quite easily just by going to the Physics department. After all, good aliens are supposed to behave in a consistent way which isn't at all understandable, and that's exactly what they do.



Approach 2: continuation maps. This approach is due to Floer. Define a vector field V on $[0,1]_t \times X$ by

$$V(t,x) = \beta(t)\frac{\partial}{\partial t} + \nabla^{g_t} f_t \ .$$

The idea is that β looks like section 1.2. Given $p_0 \in \operatorname{crit}(f_0)$ and $p_1 \in \operatorname{crit}(f_1)$ define

$$m^V(p_1,p_0) = \left\{ u : \mathbb{R} \to [0,1] \times X \mid u'(s) = V(u(s)), \quad \begin{matrix} \lim_{s\to+\infty} u(s) = (1,p_1) \\ \lim_{s\to-\infty} u(s) = (0,p_0) \end{matrix} \right\}$$

modulo reparameterization. Generically,

$$\dim m^V(p_1,p_0) = \operatorname{ind}(p_1) - \operatorname{ind}(p_0)$$

where

$$\text{“ind”}(0,p_0) = \operatorname{ind}(p_0) \qquad \text{“ind”}(1,p_1) = \operatorname{ind}(p_1) + 1 \ .$$

Define

$$\psi : C_*^{\text{Morse}}(X,f_1,g_1) \to C_*^{\text{Morse}}(X,f_0,g_0)$$

by

$$\varphi(p_1) = \sum_{\substack{p_0 \in \operatorname{crit}(f_0) \\ \operatorname{ind}(p_0) = \operatorname{ind}(p_1)}} \left[\# m^V(p_1,p_0)\right] p_0$$

where # denotes the count in $\mathbb{Z}$ with signs (or in $\mathbb{Z}/2\mathbb{Z}$).

Lemma 1. *φ is a chain map $\partial_0 \circ \varphi = \psi \circ \partial_1$ where ∂_i is the differential for (f_i, g_i).*

Proof sketch. Similarly to the proof that $\partial^2 = 0$, if $\operatorname{ind}(p_1) - \operatorname{ind}(p_0) = 1$, then $m^V(p_1,p_0)$ has a compactification $\overline{m^V(p_1,p_0)}$ is a compact 1-manifold with boundary. The idea is that we want to think about the different ways that these flow-lines can break. We have that:

$$\partial \overline{m^V(p_1,p_0)} = \coprod_{\substack{p_0' \in \operatorname{crit}(f_0) \\ \operatorname{ind}(p_0') = \operatorname{ind}(p_1)}} m^V(p_1,p_0') \times m(p_0',p_0) \qquad (1)$$

$$\amalg \coprod_{\substack{p_1' \in \operatorname{crit}(f_1) \\ \operatorname{ind}(p_1') = \operatorname{ind}(p_0)}} m(p_1,p_1') \times m^V(p_1',p_0)$$

$$
\begin{aligned}
0 = \#\partial\overline{m^V(p_1,p_0)} &= \sum_{p_0'} \#m^V(p_1,p_0')\,\#m(p_0',p_0) + \sum_{p_1'} \#m(p_1,p_1')\,\#m^V(p_1',p_0) \\
&= \sum_{p_0'} \langle \varphi p_1, p_0'\rangle \langle \partial_0 p_0', p_0\rangle + \sum_{p_1'} \langle \partial p_1, p_1'\rangle \langle \varphi p_1', p_0\rangle \\
&= \langle \partial_0 \varphi p_1, p_0\rangle + \langle \varphi \partial_1 p_1, p_0\rangle
\end{aligned}
$$

with $\mathbb{Z}/2\mathbb{Z}$ coefficients. □

Lemma 2. *The continuation map*

$$H_*^{Morse}(f_1,g_1) \to H_*^{Morse}(f_0,g_0)$$

does not depend on the choice of path $\{(f_t,g_t)\}$.

Proof. Let $\{(f_{0,t},g_{0,t})\}$ and $\{(f_{1,t},g_{1,t})\}$ be two generic paths. The space of such paths is contractible, so we can choose a generic homotopy

$$\{(f_{\tau,t},f_{\tau,t})\}_{\tau\in[0,1],t\in[0,1]}$$

between them, i.e. for all τ we have

$$(f_{\tau,0},g_{\tau,0}) = (f_0,g_0) \qquad\qquad (f_{\tau,1},g_{\tau,1}) = (f_1,g_1)\ .$$

Define a vector field W on

$$[0,1]_\tau \times [0,1]_t \times X$$

by

$$W(\tau,t,x) = \beta(t)\frac{\partial}{\partial t} + \operatorname{grad}^{g_{\tau,t}} f_{\tau,t}$$

so the flow of W preserves τ and agrees with V_τ. Now given $p_0 \in \operatorname{crit}(f_0)$ and $p_1 \in \operatorname{crit}(f_1)$, we can define

$$m^W(p_1,p_0)$$

to consist of the flow lines of W from $(\tau,0,p_0)$ to $(\tau,1,p_1)$ for some τ modulo reparameterization. Generically,

$$\dim m^W(p_1,p_0) = \operatorname{ind}(p_1) - \operatorname{ind}(p_0) + 1\ .$$

Now define

$$K : C_*^{\mathrm{Morse}}(f_1,g_1) \to C_{*+1}^{\mathrm{Morse}}(f_0,g_0)$$

by

$$K(p_1) = \sum_{\operatorname{ind}(p_0)=\operatorname{ind}(p_1)+1} \left[\#m^W(p_1,p_0)\right] p_0$$

and the claim is that this is a chain homotopy, i.e. $\partial_0 K + K\partial_1 = \varphi_0 - \varphi_1$. The idea of the proof is to suppose $\operatorname{ind}(p_0) = \operatorname{ind}(p_1)$, and then $m^W(p_1,p_0)$ has a compactification

$$\overline{m^W(p_1,p_0)}$$

whose boundary points count

$$\langle \partial_0 K p_1, p_0\rangle \qquad \langle K\partial_1 p_1, p_0\rangle \qquad \langle \varphi_0 p_1, p_0\rangle \qquad \langle \varphi_1 p_1, p_0\rangle\ .$$

The point is that breaking gives us $\partial_0 K$, $K\partial_1$, and then sending $\tau \to 0$ gives us φ_0, and $\tau \to 1$ gives us φ_1. So φ_0 and φ_1 are chain homotopic and induce the same map on Morse homology. □

Now we want to show that the continuation map $H_*^{\mathrm{Morse}}(f,g) \to H_*^{\mathrm{Morse}}(f,g)$ is the identity.

Proof. Take the constant path

Exercise 2. The vector field V is Morse-Smale and if $\operatorname{ind}(p_0) = \operatorname{ind}(p_1)$ then

$$m^V(p_1, p_0) = \begin{cases} \text{pt} & p_0 = p_1 \\ \emptyset & p_0 \neq p_1 \end{cases}.$$

[The idea is to show that the flow lines of V project to flow-lines of f.]

Therefore $\varphi : H_*^{\text{Morse}}(f, g) \circlearrowleft$ is the identity. □

The last step is the following. If (f_0, g_0), (f_1, g_1), and (f_2, g_2) are three Morse-Smale pairs, then we have the commutative triangle

$$\begin{tikzcd} H_*(f_2, g_2) \arrow[r] \arrow[rr, bend right] & H_*(f_1, g_1) \arrow[r] & H_*(f_0, g_0) \end{tikzcd}$$

and then we have a chain homotopy between these chain homotopies.

Now if we believe this, we can take $(f_2, g_2) = (f_0, g_0)$ to show that the continuation map from $H_*(f_0, g_0) \to H_*(f_1, g_1)$ is the inverse map of the continuation map in the other direction.