LECTURE 22
MATH 242

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1. Floer homology of Hamiltonian symplectomorphisms

1.1. **Morse homology.** Recall what we discussed last week. Let X be a finite dimensional smooth manifold. Then we have a Morse function $f : X \to \mathbb{R}$, the critical points of this function, the Hessian, and gradient flow lines. Then all of this data gave us Morse homology which is an invariant of X. Now we will do some sort of infinite version of this to get symplectic invariants.

1.2. **Hamiltonian Floer homology.** Let (M, ω) be a closed symplectic manifold. Consider a one-parameter family of Hamiltonians $\{H_t\}_{t\in[0,1]}$ i.e. $H_t : M \to \mathbb{R}$. WLOG let $H_1 = H_0$ and $\{H_t\}_{t\in[0,1]}$ is the restriction of a family $\{H_t\}_{t\in\mathbb{R}}$ with $H_{t+1} = H_t$.

Let $\varphi_t : M \to M$ be the time t flow. I.e. $\varphi_0 = \mathrm{id}_M$ and

$$\frac{\partial}{\partial t}\varphi_t(x) = X_{H_t}(\varphi_t(x)) \ .$$

Then we are interested in fixed points of φ_1. A useful way to think of this is that we have a natural bijection:

$$\begin{array}{ccc} \mathrm{Fix}(\varphi_1) & \longleftrightarrow & \{\gamma : \mathbb{R}/\mathbb{Z} \to M \mid \gamma'(t) = X_{H_t}(\gamma(t))\} \\ \gamma(0) & \longleftarrow & \gamma \end{array} .$$

From this point of view we are just doing Morse theory on the free loop space of M.

Defining Hamiltonian Floer homology in full generality is very complicated, so to simplify things we will make some assumptions. We will talk later about how to remove them. Assume M is *symplectically aspherical*. This means that

$$\langle c_1(TM), A\rangle = \langle [\omega], A\rangle = 0$$

for all $A \in \pi_2(M)$. E.g. $\pi_2(M) = 0$ would do the trick. Let $\mathcal{L}$ be the space of contractible smooth loops $\gamma : \mathbb{R}/\mathbb{Z} \to M$. We will be doing Morse homology on $\mathcal{L}$.

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1.3. **Symplectic action functional.** Define the *symplectic action functional* $\mathcal{A} : \mathcal{L} \to \mathbb{R}$ by

$$\boxed{\mathcal{A}(\gamma) = \int_0^1 H_t(\gamma(t))\, dt + \int_{D^2} u^*\omega}$$

where $u : D^2 \to M$ is a map with

$$u\left(e^{2\pi i t}\right) = \gamma(t) \ .$$

This disk exists because we assumed γ is contractible. This integral doesn't depend on the choice of disk because of the symplectically aspherical assumption. If we replace it with some other $u' : D^2 \to M$ then

$$\int_{D^2} u^*\omega - \int_{D^2} u'^*\omega = \int_{S^2} (u\#u')^*\,\omega = 0$$

by the assumption.

Lemma 1. *γ is a critical point of $\mathcal{A}$ iff $\gamma'(t) = X_{H_t}(\gamma(t))$ for all t.*

In particular the critical points are the same as the fixed points of φ_1 which come from contractible loops.

Proof. First notice that

$$T\gamma\mathcal{L} = \Gamma(\gamma^*TM) \ .$$

So let $\eta \in \Gamma(\gamma^*TM)$. So we have to calculate

$$\begin{aligned} d\mathcal{A}_\gamma(\eta) &= \int_0^1 (dH_t)_{\gamma(t)}(\eta(t))\, dt + \int_{S^1} \omega(\eta(t), \gamma'(t))\, dt \\ &= \int_0^1 \left(\underbrace{(dH_t)_{\gamma(t)}(\eta(t))}_{\omega\left(X_{H_t}, \eta(t)\right)} + \omega(\eta(t), \gamma'(t)) \right) dt \\ &= \int_0^1 \omega(X_{H_t} - \gamma'(t), \eta(t))\, dt \end{aligned}$$

and $\gamma \in \mathrm{crit}(\mathcal{A})$ iff this integral vanishes for all η iff $X_{H_t} - \gamma'(t) = 0$ for all t. □

Now we consider the gradient flow lines of $\mathcal{A}$. To find these we need a metric on $\mathcal{L}$. Choose an ω-compatible almost complex structure J. Then $\omega(\cdot, J)$ is a Riemannian metric on M. Now define a metric g on $\mathcal{L}$ as follows. Let $\gamma \in \mathcal{L}$ and $\eta_1, \eta_2 \in \Gamma(\gamma^*TM)$. Then we define the inner product of these to be

$$g(\eta_1, \eta_2) = \int_0^1 \omega(\eta_1(t), J\eta_2(t))\, dt \ .$$

Then the gradient flow equation is as follows. First we calculate $\nabla\mathcal{A}$. The defining property is that

$$d\mathcal{A}(\eta) = g(\nabla\mathcal{A}, \eta) \ .$$

We calculated that

$$\begin{aligned} d\mathcal{A}(\eta) &= \int_0^1 \omega\left(X_{H_t} - \gamma'(t), \eta(t)\right) dt \\ &= \int_0^1 \omega\left(J\left(X_{H_t} - \gamma'(t)\right), J\eta(t)\right) dt \\ &= g\left(J\left(X_{H_t} - \gamma'(t)\right), \eta\right) . \end{aligned}$$

Therefore

$$(\nabla\mathcal{A})_\gamma = \left(t \mapsto J\left(X_{H_t} - \gamma'(t)\right)\right) .$$

A path in $\mathcal{L}$ is a map $u : \mathbb{R}_s \times S^1_t \to M$ where $u(s,\cdot) \in \mathcal{L}$. Then a gradient flow line of $\mathcal{A}$ is a map

$$u : \mathbb{R} \times S^1 \to M$$

such that

$$\partial_s u = (\nabla\mathcal{A})(u(s,\cdot)) = J\left(X_{H_t} - \partial_t u\right)$$

so the conclusion is that

$$\boxed{\partial_s u + J\partial_t u - X_{H_t} = 0}$$

which is sometimes called "Floer's equation".

Remark 1. Note this looks quite a bit like the holomorphic curve equation, i.e. without this Hamiltonian perturbation it is a holomorphic curve.

We could also take J to depend on $t \in S^1$ and then Floer's equation is

$$\partial_s u + J_t\left(\partial_t u - X_{H_t}\right) = 0 .$$

This is often done.

We call solutions "holomorphic cylinders with time-dependent J and Hamiltonian perturbation".

Bad news: The gradient flow is not defined. Given a smooth loop γ, we generally cannot solve for $u : (-\epsilon,\epsilon) \times S^1 \to M$ satisfying Floer's equation with $u(0,\cdot) = \gamma$.

Before Floer, people didn't think it was possible to do Morse theory on this function for this (and other) reason.[1]

Good news: It still makes sense to consider moduli spaces of flow-lines between critical points.

1.4. **Moduli spaces of flow-lines.** Given $\gamma_+, \gamma_- \in \operatorname{crit}(\mathcal{A})$ define

$$\widetilde{m} = \left\{ u : \mathbb{R} \times S^1 \to M \mid \partial_s u + J_t\left(\partial_t u - X_{H_t}\right) = 0, \lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t) \right\} .$$

Now $\mathbb{R}$ acts on this set by translating the s coordinate, i.e.

$$(r \cdot u)(s,t) = u(s,t) .$$

Then define

$$m(\gamma_+,\gamma_-) = \tilde{m}(\gamma_+,\gamma_-)/\mathbb{R} .$$

[1]But then Floer said "hold my beer" and did it.

1.5. **Autonomous case.** The basic example is what is called the "autonomous case". This is where

- H_t does not depend on t and is a Morse function $H : M \to \mathbb{R}$
- J_t does not depend on t and so it defines a metric g on M.

So we have a map

$$\operatorname{crit}(H) \to \operatorname{crit}(\mathcal{L})$$

which sends p to the constant loop $\gamma(t) = p$. This is surjective possibly after multiplying H by some ϵ sufficiently small.

Let $\eta : \mathbb{R} \to M$ be a flow line of H, i.e. $\eta'(s) = \nabla H(\eta(s))$. We can calculate

$$\langle \nabla H, V \rangle = dH\ (V) = \omega(X_H, V) = \omega(\nabla H, Jv) = \omega(-J\nabla H, V)$$

to tell us that

$$\nabla H = JX_H$$

so

$$\eta'(s) = \nabla H(\eta(s)) = JX_H(\eta(s)) \ .$$

This means that $u(s,t) = \eta(s)$ solves Floer's equation since

$$\partial_s u + J(\cancel{\partial_t u} - X_{H_t}) = 0 \ .$$

Assume that all fixed points of φ are nondegenerate. In general we define the Floer chain complex $\mathrm{CF}_*(H,J)$ as follows. It is generated over $\mathbb{Z}/2\mathbb{Z}$ (or $\mathbb{Z}$) by critical points of $\mathcal{A}$. This is graded by the Conley-Zehnder[2] index which we will discuss later. In the autonomous case

$$\mathrm{cz}(\gamma \equiv p) = \operatorname{ind}(p) - n$$

where $\dim(M) = 2n$.

The differential

$$\partial : \mathrm{CF}_*(H,J) \to \mathrm{CF}_{*-1}(H,J) \ .$$

is defined to be:

$$\partial\gamma_+ = \sum_{\mathrm{cz}(\gamma_- = \mathrm{cz}(\gamma_+)-1)} \# m(\gamma_+,\gamma_-) \cdot \gamma_- \ .,$$

Now we have to prove the following facts:

- If J is generic and $\gamma_+ \neq \gamma_-$ then $m(\gamma_+,\gamma_-)$ is a smooth manifold of dimension $\mathrm{cz}(\gamma_+) - \mathrm{cz}(\gamma_-) - 1$.
- If $\mathrm{cz}(\gamma_+) - \mathrm{cz}(\gamma_-) = 1$ then $m(\gamma_+,\gamma_-)$ is compact, and hence finite.
- $\partial^2 = 0$

Then we define the Floer homology to be

$$\mathrm{HF}_*(H,J) = H_*(\mathrm{CF}_*(H,J), \partial) \ .$$

Then the main theorem to prove is:

Theorem 1. $\mathrm{HF}_*(H,J) \simeq H_{*+n}(M; \mathbb{Z}/2\mathbb{Z})$.

Now we know that the chain complex has enough generators to produce this homology, which shows the Arnold conjecture. We actually get a slightly stronger statement. In fact the number of fixed points for which the corresponding loop is contractible is at least the sum of the Betti numbers. So there's a lot of hard work to be done on this next time.

[2] This is pronounced like the German 'z' which is like an english 'ts'.

2. Fun digression

Let's go back to the finite dimensional case. Suppose we have $f : X \to \mathbb{R}$ a Morse function with g a metric on X. Suppose we now the pair (f, g) is Morse-Smale. Then $-f : X \to \mathbb{R}$ is also a Morse function. In fact we have

$$m^f(p_+, p_-) = m^{-f}(p_-, p_+)$$

and the pair $(-f, g)$ is also Morse-Smale. In particular we have an isomorphism of chain complexes[3]

$$C_*(-f, g) = C^{n-*}(f, g)$$

where $n = \dim X$. This tells us that

$$H_*(-f, g) = H^{n-*}(f, g) \ ,$$

however we know by continuation maps that in fact we have:

$$\underbrace{H_*(-f, g) \quad \simeq \quad H^{n-*}(f, g) \qquad\qquad H_*(f, g)}_{\text{continuation}}$$

and this is Poincaré duality.

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[3]At least over $\mathbb{Z}/2\mathbb{Z}$. Working over $\mathbb{Z}$ we would have to worry a bit about orientations.