LECTURE 23
MATH 242

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1. Floer theory for a Hamiltonian symplectomorphism

Let (M, ω) be a closed symplectic manifold which is symplectically aspherical, i.e. $c_1, [\omega]$ vanish on $\pi_2(M)$. Choose generic $\{J_t\}$ with $J_{t+1} = J_t$. Then we have $H_t : M \to \mathbb{R}$, $H_{t+1} = H_t$. Assume all of the fixed points of φ_1 are nondegenerate. Then consider

$$\mathcal{P}_0 = \left\{\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M \mid \gamma'(t) = X_{H_t}(\gamma(t)), \gamma \text{ contractible}\right\} \hookrightarrow \mathrm{Fix}(\varphi_1) \ .$$

Now we have $\mathrm{CF}_*(H, J)$ generated over $\mathbb{Z}/2\mathbb{Z}$ (or $\mathbb{Z}$) by $\gamma \in \mathcal{P}_0$ with $\mathrm{cz}(\gamma) = *$. For $\gamma_\pm \in \mathcal{P}_0$ we have the moduli space

$$m^J(\gamma_+, \gamma_-) = \left\{u : \mathbb{R} \times S^1 \to M \mid \partial_s u + J_t(\partial_t u - X_{H_t}) = 0, \lim_{s \to \pm\infty} \gamma_\omega(t)\right\} / \mathbb{R} \ .$$

Notice that

$$\dim m^J(\gamma_+, \gamma_-) = \mathrm{cz}(\gamma_+) - \mathrm{cz}(\gamma_-) - 1 \ .$$

Now the differential is given by:

$$\partial \gamma_+ = \sum_{\mathrm{cz}(\gamma_-) = \mathrm{cz}(\gamma_+) - 1} \# m^J(\gamma_+, \gamma_-) \cdot \gamma_- \ .$$

2. The Conley-Zehnder index

This is supposed to be an analogue of the Morse index. If $u \in \tilde{m}^J(\gamma_+, \gamma_-)$, then $T_u \tilde{m}$ is the kernel of a linearized operator

$$D_u : L_1^2(u^* TM) \to L^2(u^* TM) \ .$$

If we choose a trivialization of $u^* TM$ which converges to trivializations of $u^* TM$ which "converge to" trivializations of $\gamma_\pm^* TM$as $s \to \pm\infty$, then D_u has the form

$$D_u \xi = \partial_s \xi + J_0 \partial_t \xi + A(s, t) \xi$$

where

$$\lim_{s \to \pm\infty} A(s, t) = A_\pm(t) \ .$$

The point is that $J_0 \partial_s + A_\pm(t)$ is analogous to the negative Hessian in Morse theory.

The Morse theory analogue is as follows. A gradient flow line is a map $\eta : \mathbb{R} \to X$ such that

$$\partial_S \eta - \nabla f(\eta(s)) = 0 \ .$$

Then the linearized operator is

$$D_\eta : L_1^2(\eta^* TX) \to L^2(\eta^* TX) \ .$$

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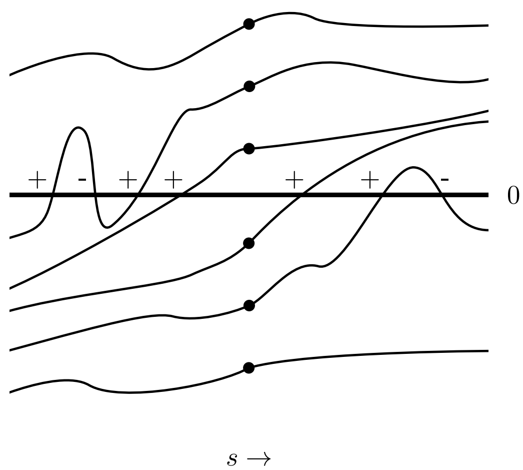

FIGURE 1. The black dots are some eigenvalues for some finite s, and the spectral flow is 3.

In a trivialization

$$D_\eta \xi = \partial_s \xi + A(s)\xi$$

and

$$\lim_{s\to\pm\infty} A(s) = -H(f, p_\pm) \ .$$

There is some slightly bad news in the Floer theory case. The operator $J_0\partial_t + A_\pm(t)$ has ∞ many positive and negative eigenvalues. So we can't do the exact same thing as the Morse theory setting.

2.1. **Spectral flow "principle".** Let H be a Hilbert space. Consider a 1-parameter family of operators $A_s : H \to H$ where

$$\text{"}\lim_{s\to\infty} A_s\text{"} = A_\pm$$

where $A_\pm$ is self-adjoint and $0 \notin \mathrm{Spec}(A_\pm)$. Under suitable technical hypotheses one defines the *spectral flow* $\mathrm{sf}(\{A_s\})$ be the "number of eigenvalues with real part crossing from negative to positive as s goes from $-\infty$ to $+\infty$ minus the number of eigenvalues that cross from positive to negative."

Let's assume all of the eigenvalues are real and draw a picture.

Theorem 1. *Under the appropriate technical hypotheses, the operator*

$$D = \partial_s + A_s : L_2^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$$

is Fredholm, and $\mathrm{ind}(D) = sf(\{A_s\})$.

Example 1 (Morse theory)**.** For $\eta : \mathbb{R} \to X$ a flow line with

$$\lim_{s\to\pm\infty} \eta(s) = p_\pm$$

then we have

$$D_\eta = \partial_s + A_d$$

where the $\lim_{s\to\pm\infty}$ is $-H(f,p_\pm)$. In this case the theorem implies that

$$\operatorname{ind}(D_\eta) = \operatorname{sf}(\{A_s\})$$

is the number of positive eigenvalues of $-H(f,p_+)$ minus the number of positive eigenvalues of $-H(f,p_-)$ which is is the number of negative eigenvalues of $H(f,p_+)$ minus the number of negative eigenvalues of $H(f,p_-)$ which is the difference of the indices as we should expect.

In the Floer theory case we have

$$D_u = \partial_s + J_0\partial_t A(s,t)$$

and the goal is to define the Conley-Zehnder index $\operatorname{cz}(\gamma_\pm)$ such that

$$\operatorname{sf}\left(\{J_0\partial_t + A(s,t)\}_s\right) = \operatorname{cz}(\gamma_+) - \operatorname{cz}(\gamma_-) \ .$$

Let $\{\psi_t\}_{t\in[0,1]}$ be a family of $2n\times 2n$ symplectic matrices with $\psi_0 = 1$ and $1\notin \operatorname{Spec}(\psi_1)$. Define $\operatorname{cz}(\{\psi_+\})\in\mathbb{Z}$ as follows. The Maslov cycle is

$$\Lambda = \{A\in \operatorname{Sp}(2n) \mid \det(A-1)=0\} \ .$$

This is a codimension 1 subvariety of $\operatorname{Sp}(2n)$.

Fact 1.
- *Λ has a canonical coorientation.*
- *$H_1(\operatorname{Sp}(2n)) \simeq \mathbb{Z}$.*
- *$[\Lambda] = 2\cdot$generator.*

Now the idea is that ψ_t is some path in $\operatorname{Sp}(2n)$, and then $\operatorname{cz}(\{\psi_t\})$ counts the intersections of the path ψ_t with the Maslov cycle. The issue is that ψ_t necessarily starts life on Λ, and in particular at a very singular point of Λ. So we need to fix this.

Definition 1. Let

$$\rho_t = \begin{pmatrix} e^t & 0 \\ \underbrace{0}_{x_1} & \underbrace{e^t}_{y_1}\end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} e^t & 0 \\ \underbrace{0}_{x_n} & \underbrace{e^{-t}}_{y_n}\end{pmatrix}$$

. Declare that $\operatorname{cz}(\{\rho_t\}) = 0$. Now given $\{\psi_t\}$ let $\{\eta_t\}$ be obtained by concatenating the reverse of $\{\rho_t\}$ with $\{\psi_t\}$, and perturbing to be transverse to Λ. Define cz to be the signed count:

$$\operatorname{cz}(\{\psi_t\}) = \#(\{\eta_t\}\cap\Lambda) \ .$$

Example 2. Let $n=1$. In this case $\operatorname{Sp}(2) = S^1\times D^2$. The Maslov cycle looks as in fig. 2.

2.2. **General properties of the Conley-Zehnder index.** If A is a symmetric matrix with $\|a\| < 2\pi$ then the following index is the signature of A:

$$\operatorname{cz}\left(\left\{e^{tJ_0A}\right\}_{t\in[0,1]}\right) = \sigma(A) \ .$$

This is the property that in the autonomous case of Floer theory $\operatorname{cz} = \operatorname{ind} - n$.

If $\varphi_t : S^1 \to \operatorname{Sp}(2n)$ is a loop with $\varphi_0 = \mathrm{id}$, then

$$\operatorname{cz}(\{\varphi_t\psi_t\}) - \operatorname{cz}(\{\psi_t\}) = 2[\varphi_t] \ .$$

This describes how cz is Floer theory depend on a choice of trivialization.

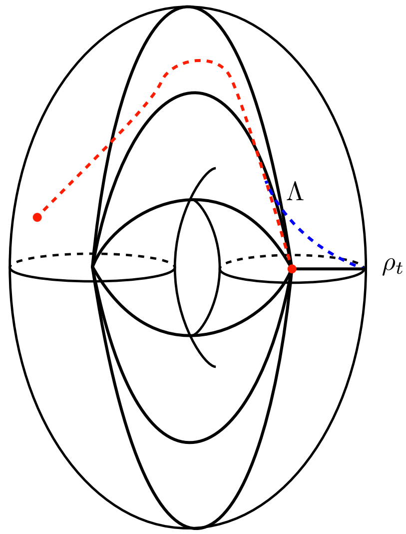

Figure 2. The entire group $\mathfrak{sp}(2)$ looks like this torus $S^1 \times D^2$. Then Λ looks like this double banana. These matrices look like rotation matrices by an angle θ. The elements inside the top banana have eigenvalues $e^{\omega i\theta}$ and correspond to a positive rotation. The elements inside the top banana also have eigenvalues $e^{\omega i\theta}$ but correspond to a negative rotation. The path $\{\psi_t\}$ in red gets changed to the one in blue by $\{\rho\}_t$. This example has $\mathrm{cz} = 1$. If it went along the bottom side instead it has $\mathrm{cz} = -1$.