LECTURE 24
MATH 242

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Professor Hutching's favorite color is ultraviolet.

Recall last time we saw that the index is the spectral flow. This means roughly that if we have

$$\partial_s + A(s) : L^2_1(\mathbb{R}, H) \to L^2(H)$$

with

$$\lim_{s\to\pm\infty} A(s)$$

self-adjoint, no kernel, then this implies that D is Fredholm and $\operatorname{ind}(D)$ is the spectral flow of $\{A_s\}$.

Example 1 (Simplest example)**.** Let $H = \mathbb{R}$ be the Hilbert space. Then

$$\lim_{s\to\pm\infty} A(s) \neq 0 \ .$$

By the fundamental theorem of ODEs, given $\psi_0 \in \mathbb{R}$, there exists unique $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = \psi_0$ and

$$\psi'(s) = -A(s)\psi(s) \ .$$

This means

$$\dim\ker D = \begin{cases} 1 & A_+ > 0, A_- < 0 \\ 0 & \text{o/w} \end{cases} \ .$$

Then the cokernel is

$$\operatorname{coker}(D) \simeq \ker(D^*) = \ker(-\partial_s + A(s))$$

so by the same argument

$$\dim\operatorname{coker} D = \begin{cases} 1 & A_+ < 0, A_- > 0 \\ 0 & \text{o/w} \end{cases} \ .$$

In the language of spectral flow, the first case is sf $= +1$, the second is sf $= 0$ or -1. In the third case sf $= -1$ and the last is sf $= 0$ or 1.

1. Hamiltonian Floer theory

1.1. The index of this operator.

Let $H : S^1 \times M \to \mathbb{R}$ and

$$\gamma_\pm : S^1 \to M \qquad\qquad \gamma'_\pm(t) = X_{H_t}(\gamma(t)) \ .$$

Then we have

$$\tilde{m}(\gamma_+, \gamma_-) = \left\{ u : \mathbb{R}\times S^1 \to M \mid \partial_s u + J_t(\partial_t u - X_{H_t}) = 0, \lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(s,t) \right\} \ .$$

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We want to understand $\dim \tilde{m}(\gamma_+, \gamma_-)$. Given u we have a linearized operator

$$D_u L : L^2_1(u^* TM) \to L^2(u^* TM) \ .$$

We can choose a trivialization of $u^* TM$ in which

$$D_u = \partial_s + J_0 \partial_t + A(s,t) \ .$$

Then we claim:

Claim 1.

$$\lim_{s \to \pm\infty} A(s,t) = A_\pm(t)$$

and

$$J_0 \partial_t + A_\pm(t) = J_0 \nabla_t^\pm$$

where $\nabla^\pm$ is the connection on $\gamma_\pm^* TM$ is given by the derivative of the flow of X_{H_t}.

We now clarify what exactly we mean. We have a flow $\varphi_t : M \to M$ (for $t \in \mathbb{R}$) with $\varphi_0 = \mathrm{id}_M$ and

$$\frac{d}{dt} \varphi_t(p) = X_{H_t}(\varphi_t(p)) \ .$$

If $\gamma : S^1 \to M$, $\gamma'(t) = X_{H_t}(\gamma(t))$ then $\varphi_t(\gamma(0)) = \gamma(t)$. Then we have a map

$$d\varphi_t : T_{\gamma(0)} M \to T_{\gamma(t)} M$$

which we can think of as parallel transport, i.e. we get a connection ∇ on $\gamma^* TM$. So for every 1-periodic orbit we get this connection.

In particular, $\ker\left(J_0 \nabla_t^\pm\right) \neq 0$ iff there exists a nonzero section η of $\gamma_\pm^* TM$ with $\nabla_t \eta = 0$, which is true iff there exists a nonzero vector $V \in T_{\gamma^\pm(0)} M$ with $d\varphi_1(V) = V$. This is equivalent to $1 \in \mathrm{Spec}\left(d\varphi_1 : T_{\gamma_\pm(0)} M \circlearrowleft\right)$ which is equivalent to the fixed point $\gamma_\pm(0)$ of φ_1 being degenerate. This means that if the fixed points of φ_1 are nondegenerate, then the operators $J_0 \nabla_t^\pm$ have kernel 0.

Note also that $A_\pm(t)$ is symmetric,[1] so $J_0 \partial_t + A_\pm(t)$ is self-adjoint, so

$$\mathrm{ind}(D_u) = \mathrm{sf}\left(\{J_0 \partial_t + A(s,t)\}_{s \in \mathbb{R}}\right)$$

Lemma 1. *Let $\{\psi_t\}_{t \in [0,1]}$ be a family of $2n \times 2n$ matrices with ψ_0 symplectic. Then ψ_t is symplectic for all t iff A_t is symmetric for all t such that $\psi_t' = J_0 A_t \psi_t$.*

Proof. ψ_t is symplectic iff $\psi_t^T J_0 \psi_t = J_0$. Now we take the derivative to get

$$\begin{aligned} 0 &= (\psi_t')^T J_0 \psi_t + \psi_t^T J_0 + \psi_t' \\ &= \cancel{\psi_t^T} A_t^T (-J_0) J_0 \cancel{\psi_t} + \cancel{\psi_t^T} J_0 J_0 A_t \cancel{\psi_t} \\ &= A_t^T - A_t = 0 \ . \end{aligned}$$

□

Define ψ_t^ω by $\psi_0^\pm = \mathrm{id}$ and

$$\frac{d}{dt} \psi_t^\pm = J_0 A_t^\pm \psi_t^\pm \ .$$

Then the matrix $\psi_t^\pm$ is symplectic because by the previous claim, it is the parallel transport map $T_{\gamma_\pm(0)} M \to T_{\gamma_\pm(t)} M$. This implies $A_t \pm$ is symmetric.

[1]We prove this below.

Proposition 1.

$$\operatorname{ind}\left(\partial_s + J_0\partial_t + A\left(s,t\right)\right) = \operatorname{cz}\left(\left\{\psi_t^+\right\}\right) - \operatorname{cz}\left(\left\{\psi_t^-\right\}\right) \ .$$

Proof. Since the index of a Fredholm operator is invariant under homotopy of Fredholm operators, WLOG $A\left(s,t\right)$ is symmetric for all s,t not just as $s \to \pm\infty$. Now define $\psi_{s,t}$ ($s \in \mathbb{R}$ and $t \in [0,1]$) by

$$\frac{\partial}{\partial t}\psi_{s,t} = A\left(s,t\right)\psi_{s,t} \ .$$

then since $A\left(s,t\right)$ is symmetric, the $\psi_{s,t}$ are symplectic.

Claim 2. For fixed s, $\ker\left(J_0\partial_t + A\left(s,t\right)\right) \neq 0$ iff $1 \in \operatorname{Spec}\left(\psi_{s,1}\right)$.

Proof. There exists nonzero $\eta \in \ker\left(J_0\partial_t + A\left(s,t\right)\right)$ iff there exists nonzero $\eta : S^1 \to \mathbb{R}^{2n}$ with $\left(J_0\partial_t + A\left(s,t\right)\right)\eta = 0$, or equivalently

$$\partial_t\eta = J_0 A\left(s,t\right)\eta \ .$$

This is equivalent to there existing nonzero $V \in \mathbb{R}^{2n}$ with $\psi_{s,1}\left(V\right) = V$. □

One can further show that the spectral flow of this family of operators

$$\left\{J_0\partial_t + A\left(s,t\right)\right\}_{s\in\mathbb{R}}$$

is the algebraic intersection number of $\left\{\psi_{s,t}\right\}_{s\in\mathbb{R}}$ with the Maslov cycle Λ. The point is that we have a homotopy $\left\{\psi_t^+\right\} \sim \left\{\psi_t^-\right\} * \left\{\psi_{s,1}\right\}$ where $*$ denotes concatenation. The idea is that we have an infinite rectangle of height 1 where the bottom edge is just the identity, the top is $\psi_{s,1}$, and the sides are ψ_t^- and ψ_t^+ respectively. Then going along the right side is the same as going along all three of the others. This implies that

$$\operatorname{cz}\left(\left\{\psi_t^+\right\}\right) = \operatorname{cz}\left(\left\{\psi_t^-\right\}\right) + \underbrace{\#\left\{\psi_{s,1}\right\} \cap \Lambda}_{\operatorname{ind}(D)} \ .$$

■

Assume all fixed points of φ are nondegenerate. In the Floer theory setup, suppose $\gamma : S^1 \to M$ satisfies $\gamma' = X_{H_t} \circ \gamma_1$.

Let τ be a symplectic trivialization of γ^*TM. Define the Conley-Zehnder index $\operatorname{cz}_\tau\left(\gamma\right) \in \mathbb{Z}$ as follows. Define a path of symplectic matrices ψ_t by $\psi_0 = \operatorname{id}$, $1 \notin \operatorname{Spec}\left(\psi_1\right)$. Then we have

$$\begin{array}{ccc} T_{\gamma(0)}M & \xrightarrow{d\varphi_t} & T_{\gamma(t)}M \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \mathbb{R}^{2n} & \overset{\psi_t}{\dashrightarrow} & \mathbb{R}^{2n} \end{array}$$

and we have

$$\boxed{\operatorname{cz}_\tau\left(\gamma\right) = \operatorname{cz}\left(\left\{\psi_t\right\}\right)} \ .$$

Remark 1. $\operatorname{cz}_\tau\left(\gamma\right)$ is invariant under homotopy of τ. The collection of homotopy classes of symplectic trivializations of γ^*TM is an affine space over $\mathbb{Z}$. Shifting τ by 1 shift cz by ± 2.

Let $u \in \tilde{m}(\gamma_+, \gamma_-)$. Let τ_+, τ_- be trivializations of $\gamma_+^* TM$, $\gamma_-^* TM$. Suppose τ_+ and τ_- extend to trivializations of $u^* TM$. Then

$$\boxed{\operatorname{ind}(D_u) = \mathrm{cz}_{\tau_+}(\gamma_+) - \mathrm{cz}_{\tau_-}(\gamma_-)} \ .$$

If D_u is surjective, then $\tilde{m}(\gamma_+, \gamma_-)$ is a manifold near u of dimension $\operatorname{ind}(D_u)$.

1.2. **Grading.** Let (M, ω) be a closed symplectic manifold. Assume (M, ω) is *symplectically aspherical*, i.e. ω and $c_1(TM)$ vanish on $\pi_2(M)$. So now we're given $H : S^1 \times M \to \mathbb{R}$ with φ_1 nondegenerate.

Let $\gamma : S^1 \to M$ be a 1-periodic contractible orbit with $\gamma' = X_{H_t} \circ \gamma$. Define $\mathrm{cz}(\gamma)$ as follows. Let $u : D^2 \to M$ with $u\left(e^{2\pi i t}\right) = \gamma(t)$, i.e. we choose a disk with boundary γ. Let τ be a trivialization of $\gamma^* TM$ that extends over $u^* TM$. This is fine since every vector bundle over a disk is trivial. Now define $\mathrm{cz}(\gamma) = \mathrm{cz}_\tau(\gamma)$.

Claim 3. This is well-defined.

Proof. First of all this depends only on u since the set of trivializations is contractible. So let $u' : D^2 \to M$ be another disk. Then τ extends over u iff τ extends over u'. So TM pulled back to the sphere formed by these two disks is trivial. □

So we now have that the Floer chain complex generated by contractible 1-periodic orbits. The grading of γ will be $\mathrm{cz}(\gamma)$. If $u \in \tilde{m}(\gamma_+, \gamma_-)$ and if D_u us surjective then $\tilde{m}(\gamma_+, \gamma_-)$ is a manifold near u of dimension

$$\mathrm{cz}(\gamma_+) - \mathrm{cz}(\gamma_-) \ .$$

Next time we will explain a bit more about the definition of Floer homology, and we will compute it when H does not depend on t.