LECTURE 25
MATH 242

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1. Lagrangian Floer homology

Again (M,ω) is a closed symplectic manifold. Assume for now that (M,ω) is *symplectically aspherical*, i.e. ω and $c_1(TM)$ vanish on $\pi_2(M)$. Then let $H : S^1 \times M \to \mathbb{R}$ be a time-dependent Hamiltonian. Assume $\varphi_1 : M \circlearrowleft$ has nondegenerate fixed points. Let

$$\mathcal{P}_0 G(H) = \left\{\gamma : S^2 \to M \mid \gamma'(t) = X_{H_t}(\gamma(t)), \gamma \text{ contractible}\right\} \hookrightarrow \operatorname{Fix}(\varphi_1) \ .$$

Then we have the action $\mathcal{A} : \mathcal{P}_0(H) \to \mathbb{R}$ defined by

$$\mathcal{A}(\gamma) = \int_0^1 H_t(\gamma(t))\, dt + \int_{D^2} u^* \omega$$

where $u : D^2 \to M$ (which exists since γ is contractible) and $u\left(e^{2\pi i t}\right) = \gamma(t)$.

We also have the Conley-Zehnder index $\mathrm{cz} : \mathcal{P}_0(H) \to \mathbb{Z}$ Given $\gamma \in \mathcal{P}_0(H)$ choose a trivialization τ of $\gamma^* TM$ which extends to the trivialization of $u^* TM$ for u as above. Then we get a path $\{\psi_t\}_{t \in [0,1]}$ in $\mathrm{Sp}(2n)$ where $\psi_0 = 1$ and $1 \notin \operatorname{Spec}(\psi_1)$, then this gives us

$$\begin{tikzcd}
T_{\gamma(0)}M \arrow[r, "d\varphi_t"] \arrow[d, "\tau"] & T_{\gamma(t)}M \arrow[d, "\tau"] \\
\mathbb{R}^{2n} \arrow[r, "\psi_t"] & \mathbb{R}^{2n}
\end{tikzcd} .$$

The index is then $\mathrm{cz}(\gamma) = \mathrm{cz}(\{\psi_t\})$.

Choose $\{J_t\}$ a generic path of ω-compatible acs. $\mathrm{CF}_*(H,J)$ generated over $\mathbb{Z}/2\mathbb{Z}$ (or $\mathbb{Z}$) by $\mathcal{P}_0(H)$. The grading is given by cz. Then

$$\langle \partial \gamma_+, \gamma_- \rangle$$

counts $u : \mathbb{R} \times S^1 \to M$ such that

$$\partial_s u + J_t(\partial_t u - X_{H_t}) = 0 \qquad\qquad \lim_{s \to \pm\infty} u(s,t) = \gamma_\pm(t)$$

modulo translation by s.

Claim 1.
- ∂ is well-defined (assuming $\{J_t\}$ is generic),
- $\partial^2 = 0$, i.e. we have a well defined $\mathrm{HF}_*(H,J)$.

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If $\operatorname{cz}(\gamma_+) - \operatorname{cz}(\gamma_-) = 2$ then we want to show that $m^J(\gamma_+, \gamma_-)$ has a compactification to a compact manifold $\overline{m^J}(\gamma_+, \gamma_-)$ with boundary

$$\partial \overline{m^J}(\gamma_+, \gamma_-) = \coprod_{\operatorname{cz}(\gamma_+) - \operatorname{cz}(\gamma_-) = 1} m^J(\gamma_+, \gamma_0) \times m^J(\gamma_0, \gamma_-) \ .$$

Remark 1. Solutions to Floer's equation are equivalent to certain J-holomorphic maps as follows.

If $\varphi : (M, \omega) \circlearrowleft$ is any symplectomorphism, define the *mapping torus*

$$Y_\varphi = [0,1] \times M / \{(1, x) \sim (0, \varphi(x))\} \ .$$

This is in fact a smooth fiber bundle

$$\begin{tikzcd} M \arrow[r] & Y_\varphi \arrow[d] \\ & S^1 = [0,1]/0 \sim 1 \end{tikzcd} \ .$$

We can cross this with $\mathbb{R}$ to get a fiber bundle:

$$\begin{tikzcd} M \arrow[r] & Y_\varphi \times \mathbb{R} \arrow[d, "\pi"] \\ & S^1 \times \mathbb{R} \end{tikzcd} \ .$$

Note that $\mathbb{R} \times Y_\varphi$ is symplectic with symplectic form

$$\pi^* (\, ds \ dt \,) + \tilde{\omega} \ .$$

Suppose $\varphi = \varphi_1$ where $\{\varphi_t\}_{t \in [0,1]}$ is a Hamiltonian isotopy where $\varphi_0 = \mathrm{id}$ and

$$\frac{d}{dt} \varphi_t = X_{H_t} \circ \varphi_t \ .$$

Then a map $u : \mathbb{R} \times S^1 \to M$ is equivalent to a section

$$\psi : \mathbb{R} \times S^1 \to \mathbb{R} \times Y_\varphi \ .$$

Given u, define

$$\psi(s,t) = \left(s, \varphi_t^{-1}(u(s,t))\right)$$

for $s \in \mathbb{R}$ and $t \in [0,1]$. The 1-parameter family $\{J_t\}_{t \in S^1}$ determines a compatible acs $\mathbb{J}$ on the bundle $\mathbb{R} \times Y_\varphi$. On $\{(,t)\} \times M$ (for $s \in \mathbb{R}$, $t \in [0,1]$)

$$\mathbb{J} = (\varphi_t)_*^{\pm} \circ J_T \circ (\varphi_t)_*^{\mp 1} \qquad\qquad \mathbb{J}\left(\tilde{\partial}_s\right) = \tilde{\partial}_t \ .$$

Exercise 1. A solution of Floer's equation is equivalent to a $\mathbb{J}$-holomorphic section $\psi : \mathbb{R} \times S^1 \to \mathbb{R} \times Y_\varphi$.

The dictionary is as follows:

$\operatorname{Fix}(\varphi_1)$	$\leftrightarrow$	Parallel sections of $\begin{array}{c} Y_\varphi \\ \downarrow \\ S^1 \end{array}$
$u : \mathbb{R} \times S^1 \to M$	$\leftrightarrow$	sections $\psi : \mathbb{R} \times S^1 \to \mathbb{R} \times Y_\varphi$
u satisfies Floer's equation	$\leftrightarrow$	ψ is $\mathbb{J}$-holomorphic

To prove the compactness needed to show that ∂ is well-defined and $\partial^2 = 0$, we need to rule out bubbling of $\mathbb{J}$-holomorphic spheres in $\mathbb{R} \times Y_\varphi$. If u is such a sphere $u : S^2 \to \mathbb{R} \times Y_\varphi$ is $\mathbb{J}$-holomorphic, then the composition

$$S^2 \to \mathbb{R} \times Y_\varphi \to \mathbb{R} \times S^1$$

is holomorphic, which means this is constant and u is a J_t-holomorphic sphere in the fiber over some (s,t). Now the symplectically aspherical assumption implies that u is constant.

Remark 2. This mapping torus point of view allows one to define Floer homology for any nondegenerate symplectomorphism, not necessarily Hamiltonian isotopic to the identity.

So we've seen that $\mathrm{HF}_*(H,J)$ is well-defined and

$$|\mathrm{Fix}(\varphi_1)| \geq \sum_i \operatorname{rank} \mathrm{HF}_i(H,J) \ .$$

Claim 2. $\mathrm{HF}_*(H,J)$ is independent of H and J.

Let $(H_0, J_{0,t})$, $(H_1, J_{1,t})$ be two generic pairs. Choose a generic homotopy $\{(H_s, J_{s,t})\}_{s\in\mathbb{R}}$ with

$$(H_s, J_{s,t}) = \begin{cases} (H_0, J_{0,t}) & s \leq 0 \\ (H_1, J_{1,t}) & \geq 1 \end{cases}$$

Given $\gamma_0 \in \mathcal{P}_0(H_0)$ and $\gamma_1 \in \mathcal{P}_0(H_1)$, consider

$$u : \mathbb{R}_s \times S^1_t \to M$$

satisfying

$$\begin{gathered} \partial_s u + J_{s,t}\left(\partial_t u - X_{H_{s,t}}\right) = 0 \\ \lim_{s\to\infty} u(s,t) = \gamma_1(t) \qquad\qquad \lim_{s\to-\infty} u(s,t) = \gamma_0(t) \ . \end{gathered} \tag{1}$$

Generically, the space of solutions is a manifold of dimension $\mathrm{cz}(\gamma_1) - \mathrm{cz}(\gamma_0)$. Define

$$\varphi : \mathrm{CF}_*(H_1, J) \to \mathrm{CF}_*(H_0, J_0)$$

by defining $\langle \varphi\gamma_1, \gamma_0 \rangle$ to be the $\pmod 2$ count of solutions to (1).

Lemma 1.

- *This is a well defined chain map.*
- *The map on homology does not depend on the homotopy from (H_0, J_0) to (H_1, J_1) which implies we have a well-defined map*
  $$\Phi : \mathrm{HF}_*(H_1, J_1) \to \mathrm{HF}_*(H_0, J_0)$$
- $$\begin{tikzcd} \mathrm{HF}_*(H_2, J_2) \arrow[r, "\Phi"] \arrow[rr, bend right, "\Phi"'] & \mathrm{HF}_*(H_1, J_1) \arrow[r, "\Phi"] & \mathrm{HF}_*(H_0, J_0) \end{tikzcd} \ .$$

The last step is then to show that $\Phi : \mathrm{HF}_*(H,J) \circlearrowleft$ is the identity. To prove this, choose

$$(H_s, J_{s,t}) = (H, J_t) \ .$$

[We skip showing that this constant homotopy satisfies the required transversality.]

Lemma 2. *If* $\mathrm{cz}(\gamma_0) = \mathrm{cz}(\gamma_1)$ *then the set of solutions to* (1) *is empty if* $\gamma_0 \neq \gamma_1$*, and a single point if* $\gamma_0 = \gamma_1$*.*

Proof. A solution to (1) is a solution to Floer's equation for (H, J). If $\gamma_0 \neq \gamma_1$ then $m^J(\gamma_1, \gamma_9) = \emptyset$ since its dimension is -1. If $\gamma_0 = \gamma_1$ then we have the "constant"constanolution

$$u(s,t) = \gamma(t) \ .$$

Then Floer's equation becomes:

$$\cancel{\partial_s u} + J_t \cancel{(\partial_t u - X_{H_t})} = 0$$

so this is a solution. And there is no other solution because if there is a solution u to Floer's equation which is non-constant (in s) with

$$\lim_{s \to \pm\infty} u(s,t) = \gamma_{\pm}(t)$$

then

$$\mathcal{A}(\gamma_+) > \mathcal{A}(\gamma_-) \ .$$

□

Theorem 1. *There is a canonical isomorphism*

$$\mathrm{HF}_*(H, J) = H^{sing}_{*+n}(M) \ .$$

This implies $\mathrm{rank}(\mathrm{HF}_i) = b_{i+n}$ which implies the Arnold conjecture.

Proof punchline. The idea is to choose $H : M \to \mathbb{R}$ a Morse function and take $H_t = H$. Then choose J ω-compatible such that the associated metric g makes the pair (H, g) Morse-Smale. Choose $J_t = J$. Then we show that if $\epsilon > 0$ is sufficiently small, then $\mathrm{CF}(\epsilon H, J)$ is canonically isomorphic to the Morse complex:

$$\mathrm{CF}_*(\epsilon H, J) = C^{\mathrm{Morse}}_{*+n}(\epsilon H, g)$$

and then we use the fact that Morse homology agrees with singular homology. □