LECTURE 26
MATH 242

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1. Symplectically aspherical case

Let (M,ω) be a closed symplectic manifold which is symplectically aspherical, so $c_1(TM)$ and $[\omega]$ vanish on $\pi_2(M)$.

Given generic $H : S^1 \times M \to \mathbb{R}$ and a generic family $J = \{J_t\}$, $\mathrm{HF}_*(H,J)$ is well-defined and depends only on M.

Choose $H : M \to \mathbb{R}$ a Morse function and an ω-compatible J so we have that if

$$g(V,W) = \omega(V,JW)$$

then (H,g) is Morse-Smale.

Proposition 1. *If $\epsilon > 0$ is sufficiently small then there is an isomorphism of chain complexes*

$$\mathrm{CF}_*(\epsilon H, J) = C^{Morse}_{*+n}(H,g) \otimes \mathbb{Z}/2\mathbb{Z} \ .$$

Proof. There is an obvious inclusion:

$$\mathrm{crit}(H) \hookrightarrow \mathcal{P}_0(H) = \left\{\gamma : S^1 \to M \text{ contractible}, \gamma' = X_H \circ \gamma\right\} \ .$$

Then the following is a classical lemma that we will assume:

Lemma 1. *This is a bijection if $\epsilon > 0$ is small enough.*

It is also true that $\mathrm{cz}(p) = \mathrm{ind}(p) - n$.

Let $\eta : \mathbb{R} \to M$ be a flow line of ∇_H. If

$$\eta'(s) = \nabla H(\eta(s)) \qquad \lim_{s\to\pm\infty} \eta(s) = p_\pm \in \mathrm{crit}(H)$$

then $u : \mathbb{R} \times S^1 \to M$ defined by

$$u(s,t) = \eta(s)$$

is a solution to Floer's equation.

So we have an inclusion

$$m^{\mathrm{Morse}}(p_+,p_-) \hookrightarrow m^{\mathrm{Floer}}(p_+,p_-) \ .$$

Then we want to show that if $\mathrm{ind}(p_+) - \mathrm{ind}(p_-) = 1$ and $\epsilon > 0$ is small enough then this is a bijection, and $m^{\mathrm{Floer}}(p_+,p_-)$ is cut out transversely. so if we prove this then the differentials agree.

We have an operator

$$D_\eta : L^2_1(\mathbb{R}, \eta^*TM) - L^2(\mathbb{R}, \eta^*TM) \ ,$$

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which in some trivialization looks like

$$D_\eta \xi = \partial_s \xi + A(s)\xi \ .$$

On the other hand we have

$$D_u : LL_1^2\left(\mathbb{R}\times S^1, u^*TM\right) \to L^2\left(\mathbb{R}\times S^1, u^*TM\right)$$

which in a trivialization is

$$D_u \xi = \partial_s \xi + J\partial_t \xi + A(s)\xi \ .$$

Lemma 2. *If* $\operatorname{ind}(p_+) - \operatorname{ind}(p_-) = 1$ *and if* $\epsilon > 0$ *is small enough then every element of* $\ker(D_u)$ *is* S^1*-invariant.*

Proof. Let $\xi \in \ker(D_i u)$. Use the trivialization to regard $\xi : \mathbb{R} \to \mathbb{R}^{2n}$. Let

$$\eta(s) = \int_{S^1} \xi(s,t)\ dt \ .$$

Then $(s,t) \mapsto \eta(s)$ is an element of $\ker D_u$.

So we can subtract this from ξ to get another element of $\ker D_u$. Thus it is enough to assume

$$\int_{S^1} \xi(s,t)\ dt = 0$$

for all s and prove that $\xi \equiv 0$ (if $\epsilon > 0$ is small enough).

Now we have

$$\begin{aligned}\xi(s,t) &= \int \left(\xi(s,\alpha) + \int_\alpha^t \partial_t \xi(s,\beta)\ d\beta\right) d\alpha \\ &= \int \left|\partial_t \xi(s,t)\right|\ dt\end{aligned}$$

and

$$\left|\xi(s,t)\right|^2 \leq \int \left|\partial_t \xi(s,t)\right|^2\ d\tau \ .$$

Now by a calculation in Salamon-Zehnder from 1992, we have

$$\begin{aligned}\int \left|\xi(s,t)\right|^2\ ds\ dt &\leq \int_{\mathbb{R}\times S^1} \left|\langle \xi, \partial_s \xi + J\partial_t \xi\rangle\right|^2\ ds\ dt \\ &= \int_{\mathbb{R}\times S^1} \left|\langle \xi(s,t), A(s)\rangle\right|^2\ ds\ dt \\ &\leq c\int \left|\xi(s,t)\right|^2\ ds\ dt\end{aligned}$$

If $c < 1$ then we are done. If $c \geq 1$ then multiplying H by ϵ has the effect of multiplying $A(s)$ by ϵ.

Up to $\mathbb{R}$ translation, only finitely many flow lines with $\operatorname{ind}(p_+) - \operatorname{ind}(p_-) = 1$. Therefore we can choose $\epsilon > 0$ small enough to work for all of them. □

This lemma implies that for $\epsilon > 0$ small enough the Floer trajectories coming from Morse flow-lines with index difference 1 are cut out transversely.

To complete the proof, we will show that if H has been multiplied by $\epsilon > 0$ small enough as above, $\operatorname{ind}(p_+) - \operatorname{ind}(p_-) - 1$, and N is a sufficiently large positive integer, then every Floer trajectory from p_+ to p_- for $(H/N, J)$ comes from a Morse flow line.

Proceed by contradiction. Suppose there are integers $\{N_k\}_{K=1}$ with $N_k \to \infty$ and non-S^1-invariant Floer trajectories u_k in $m^{\text{Floer}}(p_+, p_-)$ for $(H/N_k, J)$. Define $\hat{u}_k : \mathbb{R} \times S^1 \to M$ by

$$\hat{u}_k(s,t) = u_k(N_k s, N_k t) \ .$$

Then $\hat{u}_k$ satisfies Floer's equation for (H, J) since

$$\partial_s u_k + J\partial_t u_k + \frac{1}{N_k} X_H = 0 \qquad \Longrightarrow \qquad \partial_s \hat{u}_k + J\partial_t \hat{u}_k + X_H = 0 \ .$$

By compactness we can pass to a subsequence such that

$$\hat{u}_k \xrightarrow{k\to\infty} u_\infty \in m^{\text{Floer}}(p_+, p_-)$$

but now u_∞ must be S^1-invariant. By the lemma, u_∞ is isolated in $m^{\text{Floer}}(p_+, p_-)/\mathbb{R}$. Consequently u_∞ cannot be a limit of non-S^1-invariant solutions. This is a contradiction as desired. ∎

So modulo some technical things we have proven the Arnold conjecture in the symplectically aspherical case.

2. Monotone case

Up until now we have assumed that $c_1(TM)$ and $[\omega]$ vanish on $\pi_2(M)$. We can instead take the monotone case, i.e. that $c_1(TM) = \lambda[\omega]$ on $\pi_2(M)$ for some $\lambda > 0$.

2.1. Grading.

In this case HF_* no longer has an integer grading. The grading has values in $\mathbb{Z}/N$ where

$$N = 2\min\{\text{positive values of } c_1(TM) \text{ on } \pi_2(M)\} \ .$$

This is sometimes called the minimal Chern number.

Given $\gamma \in \mathcal{P}_0(H)$ we choose $u : D^2 \to M$ such that $u\left(e^{2\pi i t}\right) = \gamma(t)$. Let τ be a trivialization of $\gamma^* TM$ that extends over D^2. Define $\text{cz}(\gamma) = \text{cz}_\tau(\gamma)$. Choosing a different u shifts the RHS by a multiple of $2N$. Then we have

$$\text{cz}_\tau(\gamma) - \text{cz}_{\tau'}(\gamma) = \pm 2\langle c_1(TM), u\# u'\rangle$$

which implies

$$\text{cz}(\gamma) \in \mathbb{Z}/2N$$

is well-defined.

In the monotone case, when we prove compactness, bubbling can happen, but it has a cost.[1] As in fig. 1 c_1 goes down by $2k$, this cylinder had ind ≤ -1, so it doesn't exist for generic J.

The rest of the argument goes through as before which means in the monotone case we have

$$\text{HF}_*(H, J) \simeq \bigoplus_{k-n\equiv * \bmod 2N} H_k(M) \ .$$

This implies that

$$\sum_{*\in\mathbb{Z}/2N} \text{rank}\,\text{HF}_* = \sum_k b_j(M)$$

which still implies the Arnold conjecture.

[1] And this cost is just too much.

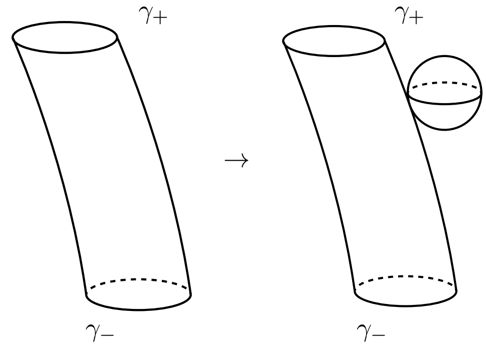

FIGURE 1. $\operatorname{ind}(u) = 1$, and when bubbling occurs we get that $\omega > 0$ which implies $c_1 = k > 0$.

3. More general cases

There are some more general cases where we need to work a bit harder. One change that needs to be made is that we take coefficients in what's called a Novikov ring.[2] Bubbling can't be ruled out in general, so to get well-defined counts of things we need virtual techniques.

[2]This is something similar to a power series ring.