LECTURE 27
MATH 242

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Today we will do an introduction to some of the other kinds of important Floer homology.

- Morse homology (model case)
- Floer homology of (Hamiltonian) symplectomorphisms (modulo Hamiltonian isotopy)
- Lagrangian Floer homology
- Fukaya category
- Heegaard Floer homology (is isomorphic to...)
- Seiberg-Witten Floer homology (is isomorphic to...)
- Embedded contact homology
- Cylindrical contact homology

1. Lagrangian Floer homology

Let (M,ω) be a closed symplectic manifold, and $L_0, L_1 \subseteq M$ closed Lagrangian submanifolds which intersect transversely. The simplest case is when $\pi_2(M, L_0) = 0$. Also assume L_0 is Hamiltonian isotopic to L_1. Define $\mathrm{CF}_*(L_0, L_1)$ to be generated (over $\mathbb{Z}/2\mathbb{Z}$) by the intersection points of L_0 and L_1. There is a relative grading by the Maslov index.[1] To define the differential, choose a generic 1-parameter family $\{J_t\}_{t\in[0,1]}$ of ω-compatible acs. If $p_-, p_+ \in L_0 \cap L_1$, define

$$\tilde{m}^J(p_+, p_-) = \left\{ u : \mathbb{R}_s \times [0,1]_t \to M \;\middle|\; \begin{array}{c} \partial_s u + J_t \partial_t u = 0 \\ \forall s, u(s,0) \in L_0, u(s,1) \in L_1 \\ \forall t, \lim_{s\to\pm\infty} u(s,t) = p_\pm \end{array} \right\}$$

The picture is as in fig. 1.

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[1] We will come back to what this means.

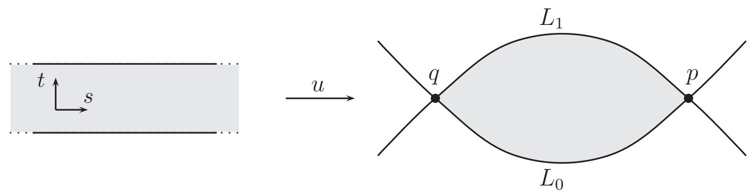

FIGURE 1. The map u maps $\mathbb{R} \times [0,1]$ to the disk on the right.

$\mathbb{R}$ acts on $\tilde{m}^J(p_+, p_-)$ by translating s. Define

$$m^J(p_+, p_-) = \tilde{m}^J(p_+, p_-)/\mathbb{R} \ .$$

Now we have that

$$\dim\left(m_u^J(p_+, p_-)\right) = (\text{Maslov index of } p_+) - (\text{Maslov index of } p_-)$$

relative to u. The differential ∂ counts these when $\dim\left(m_u^J(p_+, p_-)\right) = 0$. Then we can define $\mathrm{HF}_*(L_0, L_1)$.

Theorem 1. *Up to grading shifts, this is isomorphic to the ordinary homology $H_*(L_0)$.*

Idea. First we want to prove it is invariant under Hamiltonian isotopy of L_0 and L_1 separately. □

To compute this, start with L_0. Recall that a neighborhood of L_0 is symplectomorphic to a neighborhood of the zero section in T^*L_0. If $f : L_0 \to \mathbb{R}$ then the graph of df in T^*L_0 is Hamiltonian isotopic to the zero section. Assume f is Morse, take L_1 to be the graph of df. Multiply f by $\epsilon > 0$ as needed. Then $L_0 \cap L_1 = \mathrm{crit}(f)$. Then we want to show that ∂ on $\mathrm{CF}_*(L_0, L_1)$ agrees with the Morse differential.

Corollary 1. *If (M, ω) is a closed symplectic manifold and $L \subset M$ is a closed Lagrangian submanifold with $\pi_2(M, L) = 0$, then L is displaceable.*

Warning 1. This is false in general without the assumption that $\pi_2(M, L) = 0$.

Counterexample 1. Consider S^2 and take L to be a circle which does not divide S^2 into pieces of equal area. Then the Lagrangian Floer homology is not even defined.

Remark 1. If $\varphi : (M, \omega) \circlearrowleft$ is a Hamiltonian symplectomorphism we can define

$$L_0 = \Delta = \{(x, x) \mid x \in M\} \subset (M \times M, -\omega \oplus \omega)$$
$$L_1 = \Gamma(\varphi) = \{(x, \varphi(x)) \mid x \in M\} \subset (M \times M, -\omega \oplus \omega)$$

and then L_0 and L_1 are in fact Lagrangians. Then $L_0 \cap L_1$ consists exactly of the fixed points of φ. For suitable J, the differentials on $\mathrm{CF}_*(L_0, L_1)$ and $\mathrm{CF}_*(\varphi)$ agree.

1.1. **Fukaya category.** The objects of the Fukaya category are Lagrangians and the morphism space is exactly the Lagrangian Floer homology. Then the composition somehow counts holomorphic triangles to send:

$$\mathrm{HF}_*(L_0, L_1) \otimes \mathrm{HF}_*(L_1, L_2) \to \mathrm{HF}_*(L_0, L_2) \ .$$

2. Cylindrical contact homology

Suppose $Y \subseteq \mathbb{R}^{2n}$ is a smooth star-shaped hypersurface. (Suppose further the simplifying assumption that Y is the boundary of a convex domain.) Recall that

$$\lambda = \frac{1}{2}\sum_{i=1}^{n} (x_i\, dy_i - y_i\, dx_i)|_Y$$

is a contact form. Assume that $\lambda|_Y$ is nondegenerate. Write $\xi = \ker\lambda$.

Define the cylindrical contact homology $\mathrm{CH}_*(Y)$ as follows. $\mathrm{CC}_*(Y)$ is generated over $\mathbb{Q}$ by "good" Reef orbits. To define the differential (modulo transversality

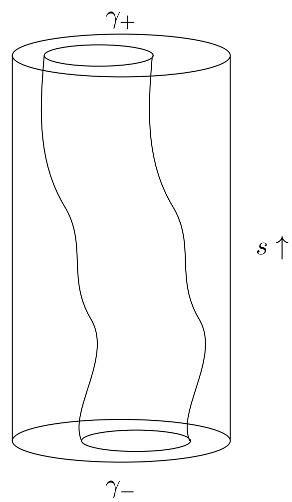

FIGURE 2. The cylinder $Y \times \mathbb{R}$ with one Reeb orbit at each end.

trouble which is okay when $n = 2$) choose a generic almost complex structure J on $\mathbb{R}_s \times Y$ such that

- $J(\partial_s) = \mathbb{R}$
- $J(\xi) = \xi$, compatibly with $d\lambda$.
- J is invariant under the $\mathbb{R}$ action translating s.

If γ_+, γ_- are Reeb orbits, define

$$\tilde{m}^J(\gamma_+, \gamma_-) = \left\{ u : \mathbb{R} \times S^1 \to \mathbb{R} \times Y \;\middle|\; \begin{array}{c} \partial_s u + J\partial_t u = 0, \\ \lim_{s \to \pm\infty} \pi_{\mathbb{R}}(u(s,t)) = \pm\infty, \\ \lim_{s \to \pm\infty} \pi_Y(u(s,t)) \text{ param. } \gamma_\pm \end{array} \right\} / \mathbb{R} \times S^1 ,$$

where this $\mathbb{R}$ is in the domain, and similarly

$$m^J(\gamma_+, \gamma_-) = \tilde{m}^J(\gamma_+, \gamma_-)/\mathbb{R}$$

where this $\mathbb{R}$ comes from the target. The picture is as in fig. 2.

If transversality holds (true for generic J when $n = 2$) then

$$\dim \tilde{m}^J(\gamma_+, \gamma_-) = \mathrm{cz}(\gamma_+) - \mathrm{cz}(\gamma_-) \ .$$

Choose a trivialization τ of ξ over Y. Given a Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ and given $t \in [0, T]$ we have

$$\begin{array}{ccc} \xi_{\gamma(0)} & \xrightarrow{\text{Lin. Reeb flow}} & \xi_{\gamma(t)} \\ \downarrow & & \downarrow \tau \\ \mathbb{R}^{2n} & \xrightarrow{\varphi_t} & \mathbb{R}^{2n} \end{array}$$

where

$$\mathrm{cz}(\gamma) = \mathrm{cz}\left(\{\varphi_t\}_{t \in [0,T]}\right) \ .$$

A Reeb orbit of γ is *bad* if it is an even multiple cover of a Reeb orbit $\overline{\gamma}$ such that $\mathrm{cz}(\gamma)$ and $\mathrm{cz}(\overline{\gamma})$ have opposite parity. Then

$$\mathrm{CC}_*(Y) = \mathbb{Q}\{\text{good Reeb orbits with } \mathrm{cz} = *\} \ .$$

The differential is defined as

$$\partial \gamma_+ = \sum_{\mathrm{cz}(\gamma_-)=\mathrm{cz}(\gamma_+)-1} \sum_{u \in m^J(\gamma_+,\gamma_-)} \epsilon(u)(\#)\gamma_-$$

where ϵ is the sign in ± 1 and $\#$ is come combinatorial factor in $\mathbb{N}^{>0}$.

Theorem 2. *$\partial^2 = 0$ and in fact*

$$\mathrm{CH}_*(Y) = \begin{cases} \mathbb{Q} & * = n-1+2n, k \in \mathbb{N}^{>0} \\ 0 & o/w \end{cases} .$$

Exercise 1. If $Y = \partial E(a_1, \ldots, a_n)$ with $a_i/a_j \notin \mathbb{Q}$ for $i \neq j$ then

$$\mathrm{cz} : \{\text{Reeb orbits}\} \xrightarrow{\text{Bij.}} \left\{ n-1+2k \,\middle|\, k \in \mathbb{N}^{>0} \right\} .$$

2.1. **Application to symplectic embeddings.** Given Y as above with $Y = \partial X$ and a positive integer k, define $c_k(X)$ to be the min of $L \in \mathbb{R}$ such that the grading $n-1+2n$ class in CH_* is represented by a linear combination of good Reeb orbits with period $\leq L$.

Example 1. $c_k(E(a_1, \ldots, a_n))$ is the kth number in the sequence of positive integer multiples of $a_1, \ldots, a_n$ written in non-decreasing order.

Theorem 3. *If there exists a symplectic embedding $X \overset{\varphi}{\hookrightarrow} X'$ then $c_k(X) \leq c_k(X')$ for all k.*

The idea of the theorem is as follows. $\varphi(X)$ sits inside X so we can define $W = X' \setminus \mathrm{int}(\varphi(X))$ which is a symplectic cobordism between ∂X and $\partial X'$ then we would have to show this induces a map on cylindrical contact homology which gives us the result.