LECTURE 3

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Recall last week we introduced the basic notions of a symplectic manifold, a Hamiltonian vector field, and a symplectomorphism. Today we will talk about symplectic embeddings and Lagrangian submanifolds.

1. Symplectic embeddings

Definition 1. Let $\left(M^{2n},\omega\right)$ and $\left(M'^{2n},\omega'\right)$ be two (not necessarily closed/compact) symplectic manifolds of the same dimension. A symplectic embedding $\varphi : (M,\omega) \hookrightarrow (M',\omega')$ is a smooth embedding $\varphi : M \hookrightarrow M'$ such that $\varphi^*\omega' = \omega$.

This is like a symplectomorphism, only it doesn't have to be surjective. A basic question we might ask is: for which (M,ω) and (M',ω') does a symplectic embedding exist? Observe there is a very basic necessary condition: if there exists a symplectic embedding $\varphi : (M,\omega) \hookrightarrow (M',\omega')$, then

$$\mathrm{Vol}\left(M^{2n},\omega\right) \le \mathrm{Vol}\left(M'^{2n},\omega\right)$$

where

$$\mathrm{Vol}\left(M^{2n},\omega\right) := \frac{1}{n!}\int_M \omega^n \ .$$

Note that this agrees with the usual definition of volume for $M \subseteq \mathbb{R}^{2n}$ and the standard symplectic form:

$$\omega_{\mathrm{std}} = \sum_{i=1}^{n} dx\ dy \qquad\qquad \omega_{\mathrm{std}}^n = n!\, dx_1\ dy_1 \ \cdots\ dx_n\ dy_n \ .$$

To see that this condition holds on the volume, we can write:

$$\begin{aligned}
\mathrm{Vol}\,(M,\omega) &= \frac{1}{n!}\int_M \omega^n = \frac{1}{n!}\int_M \left(\varphi^*\omega'\right)^n \\
&= \frac{1}{n!}\int_{\varphi(M)} \left(\omega'\right)^n = \mathrm{Vol}\left(\varphi\left(M\right),\, \omega'|_{\varphi(M)}\right) \\
&\le \mathrm{Vol}\,(M',\omega')
\end{aligned}$$

Fact 1. *This condition is sufficient for contractible open subsets of $\mathbb{R}^2$.*

So in $\mathbb{R}^2$ we just need to check that the area of one is smaller than the other in order for there to exist a symplectic embedding of one in the other. In 1985 Gromov wrote *Pseudo-holomorphic curves and symplectic manifolds* which launched the modern era of symplectic geometry. One of the first theorems shows us that the analogous statement is false in $\mathbb{R}^{2n}$ for $n > 1$. Fix $n > 1$. For $r > 0$, define the ball to be

$$B\,(r) = \left\{ z \in \mathbb{C}^n \mid \pi\,|z|^2 < r \right\} \ .$$

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So this is a ball with 2-dimensional cross section of area r. Define the cylinder to be

$$Z(r) = \left\{ z \in \mathbb{C}^n \mid \pi |z_1|^2 < r \right\} .$$

So this is a disk of area r crossed with $\mathbb{C}^{n-1}$. Then we have Gromov's nonsqueezing theorem:

Theorem 1 (Gromov)**.** *There exists a symplectic embedding $B(r) \hookrightarrow Z(R)$ iff $r \leq R$.*

Note that the direction $\Longleftarrow$ is trivial, since then the ball is just a subset of the cylinder. Also note that in the volume preserving world we could just make the sphere thinner in the appropriate direction and stretch it sufficiently in the other direction. An equivalent version of this is the following. Let $\rho : \mathbb{C}^n \to \mathbb{C}$ send $z \mapsto z_1$. Then for any symplectic embedding $\varphi : B(r) \to \mathbb{C}^n$ we have that

$$\operatorname{Area}(\rho(\varphi(B))) \geq r .$$

So the shadow can't live in a disk of area smaller than r. We can think of this as a classical version of the uncertainty principle. The z_1 factor has coordinates x_1 and y_1, and no matter what transformation we come up with to make one of the coordinates very small, the other coordinate will get much larger to conserve the area.

The following is a related result. In n dimensions, given a set of n numbers $a_1, \cdots, a_n > 0$, we can define the ellipsoid

$$E(a_1, \cdots, a_n) = \left\{ z \in \mathbb{C}^n \mid \pi \sum_{i=1}^{n} \frac{|z_i|^2}{a_i} < 1 \right\} .$$

Note that if $a_i = a$ for all i, this is just the ball of radius a.

Now we have the question of when there exists a symplectic embedding

$$E(a_1, \cdots, a_n) \hookrightarrow E(b_1, \cdots, b_n) .$$

For $n = 1$ it just has to be $a_1 \leq b_1$ for an embedding to exist. For $n = 2$ as we will see, and for $n > 2$ it is an open question.

For $n = 2$, given $a_1, a_2 > 0$ define $N(a_1, a_2)_{k \geq 0}$ to be the sequence of all linear combinations $m_1 a_1 + m_2 a_2$ where $m_i \in \mathbb{N}$ in increasing order with repetitions. For example,

$$\begin{aligned} N(1,1)_{k\geq 0} &= \{0, 1, 1, 2, 2, 2, 3, 3, 3, 3, \cdots\} \\ N(1,2)_{k\geq 0} &= \{0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, \cdots\} . \end{aligned}$$

Then there is a theorem:

Theorem 2 (McDuff,2010)**.** *There exists a symplectic embedding $E(a_1, a_2) \hookrightarrow E(b_1, b_2)$ iff*

$$N_k(a_1, a_2) \leq N_k(b_1, b_2)$$

for all k.

Remark 1. Note that:

$$\operatorname{Vol} E(a, b) = ab/2 . \tag{1}$$

Remark 2. There exists an embedding $E(1, 2) \hookrightarrow E(c, c)$ iff $c \geq 2$.

Proof. If $c \geq 2$ then $E(1,2) \subset E(c,c)$. Now we can calculate:

$$N(1,2) = (0,1,2,2,3,3,\cdots)$$
$$N(c,c) = (0,c,c,2c,2c,2c,\cdots)$$

and the result follows from the above theorem. □

McDuff-Schlenk explicitly computed, for $a \geq 1$, the function

$$f(a) = \inf\{c \mid E(1,a) \hookrightarrow B(c)\} \ .$$

The answer turns out to be very crazy. Note that f is bounded above by a, and the volume constraint in (1) implies that $f(a) \geq \sqrt{a}$. At first, f is the Fibonacci staircase until $\left(\left(1+\sqrt{5}\right)/2\right)^4$ where it is 0 until $(17/6)^2$ after which $f = \sqrt{a}$.

Exercise 1. Show that:

$$\lim_{k\to\infty} \frac{N_k(a,b)^2}{k} = 2ab = 4\,\mathrm{Vol}\,(E(a,b)) \ .$$

[Hint: It has to do with counting lattice points in triangles.]

Remark 3. Gromov non-squeezing implies that if a symplectic embedding $E(a_1,\cdots,a_n) \to E(b_1,\cdots,b_n)$ exists, then $\min(a_i) \leq \min(b_i)$.

2. Lagrangian submanifolds

Definition 2. Let $\left(M^{2n},\omega\right)$ be a symplectic manifold. A *Lagrangian submanifold* is a submanifold $L \subset M$ such that $\dim L = n$, and $\omega|_L \equiv 0$.

Remark 4. If $Z \subset M$ is a submanifold of M such that $\omega|_Z \equiv 0$, then $\dim Z \leq n$.

Proof. Let $p \in Z$. Define the symplectic complement to be

$$(T_pZ)^\omega = \{v \in T_pM \mid \forall w \in T_pZ, \omega(v,w) = 0\} \ .$$

Since $\omega : T_pM \xrightarrow{\sim} T_p^*M$ is an isomorphism, it follows that $\dim T_pZ + \dim (T_pZ)^\omega = 2n$.

Then if $\dim Z > n$, we have that $\dim (T_pZ)^\omega < n$ so there exists $v \in T_pZ \setminus (T_pZ)^\omega$ so there exists $w \in T_pZ$ with $\omega(v,w) \neq 0$, which is a contradiction. □

2.1. Examples.

Example 1. For $n = 1$, any 1-dimensional submanifold is Lagrangian.

Example 2. In $\mathbb{C}^n$, if $\gamma_1,\cdots,\gamma_n$ are simple closed curves in $\mathbb{C}$, then $\gamma_1 \times \cdots \times \gamma_n \subset \mathbb{C}^n$ is a Lagrangian submanifold diffeomorphic to $T^n = \left(S^1\right)^n$. This is Lagrangian because the cartesian product of a collection of Lagrangians is Lagrangian.

Example 3. Let X be a smooth n-dimensional manifold. In $(T^*X, d\lambda)$ we have the following Lagrangians. First we have the zero section, which is a Lagrangian since $\lambda|_X \equiv 0$. This is compact when X is compact. Any fiber F is Lagrangian since $\lambda|_F \equiv 0$. Let μ be a 1-form, and let L be the graph of μ.

Claim 1. Let $s : X \xrightarrow{\sim} L \subset T^*X$ map $p \mapsto (p,\mu(p))$. Then $s^*\lambda = \mu$.

The definition of λ implies this tautologically.[1] It follows from this claim, that $s^*\,d\lambda = d\mu$. Then L is Lagrangian iff $s^*\,d\lambda$ is identically 0, which is the case iff $d\mu = 0$, i.e. μ is closed. So a graph of a 1-form is a Lagrangian iff the 1-form is closed.

[1]Professor Hutchings says that if this is unclear, you should try to enter a state of zen.

2.2. **Exact Lagrangians.**

Definition 3. A Lagrangian $L \subset T^*X$ is exact if $L|_\lambda$ is exact.

Example 4. The graph of a 1-form μ is exact iff μ is an exact 1-form.

Conjecture 1 (Arnold's Nearby Lagrangian Conjecture)**.** *If X is a compact, connected smooth manifold and L is a compact, connected exact Lagrangian in T^*X, then L is Hamiltonian isotopic to the zero section.*

This is known for S^1 and S^2.