LECTURE 4
MATH 242

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1. Exact Lagrangians

Recall a Lagrangian submanifold in a symplectic manifold $\left(M^{2n},\omega\right)$ is an n-dimensional submanifold $L \subset M$ such that $\omega|_L = 0$. There is a related notion of an exact (with respect to λ) Lagrangian which is that $\lambda|_L = df$ for some $f\colon L \to \mathbb{R}$.

Conjecture 1 (Nearby Lagrangian conjecture)**.** *If X is a compact smooth n-manifold then every compact exact Lagrangian in T^*X is Hamiltonian isotopic to the 0-section.*

There is a related theorem of Gromov:

Theorem 1 (Gromov)**.** *There does not exist a compact exact Lagrangian in $\mathbb{R}^{2n}$.*

Since $H_1\left(\mathbb{R}^{2n}\right) = 0$, the definition of "exact Lagrangian" is independent of the choice of the primitive λ.

If $L \subseteq \mathbb{R}^{2n}$ is Lagrangian, we can define a map $\alpha\colon H_1(L) \to \mathbb{R}$ as follows. If γ is an oriented loop in L, then

$$\alpha(\gamma) = \int_{D^2} \varphi^*\omega \tag{1}$$

where $\varphi\colon D^2 \to \mathbb{R}^{2n}$ with $\varphi|_{\partial D} = \gamma$. Then L is exact iff $\alpha = 0$.

Example 1. If $\gamma_1, \cdots, \gamma_n$ are embedded loops in $\mathbb{C}$, then $L = \gamma_1 \times \cdots \times \gamma_n \subseteq \mathbb{C}^n$ is Lagrangian but not exact, because α sends each of the γ_i to the area that it encloses in $\mathbb{C}$.

2. Intersections of Lagrangians

Definition 1. A Lagrangian $L \subset M$ is displaceable if there is a Hamiltonian isotopy from L to L' such that $L \cap L' = \emptyset$.

Example 2. Suppose $M = S^2$. A Lagrangian in this manifold is given by a simple closed curve. A Hamiltonian isotopy induces a symplectomorphism, which has to conserve area. So the curve is displaceable iff it divides S^2 into two components of unequal area.

Example 3. Consider a Lagrangian in $M = T^2$, i.e. a simple closed curve. If the homology class of L is trivial, and cuts the torus into pieces of unequal area, then it is displaceable. If the homology class $[L] \neq 0 \in H_1\left(T^2\right)$ then L is not displaceable. The reason for this is as follows.

Claim 1. If $\{\varphi_t\}$ is a Hamiltonian isotopy from id to φ, then the area swept out by $\{\varphi_t(L)\}$ is zero i.e.

$$\int_{L\times[0,1]} \Phi^*\omega = 0 \tag{2}$$

for $\Phi\colon L\times[0,1]\to T^2$ sending $(x,t)\mapsto\varphi_t(x)$.

Proof. Exercise. □

If L is displaceable, then the region between L and the displaced Lagrangian L' has area area$(U)\in\mathbb{Z}$, but area$\left(T^2\right)=1$ so we have a contradiction.[1]

Remark 1. Later we will see that there is a much more systematic way to understand the question of when Lagrangians intersect using Lagrangian Floer homology, which produces an invariant of a pair of Lagrangians up to Hamiltonian isotopy.

Example 4. If $\varphi\colon(M,\omega)\to(M,\omega)$ then the graph

$$\Gamma(\varphi)=\{(x,\varphi(x))\}\subset(M\times M,(-\omega,\omega)) \tag{3}$$

is a Lagrangian. An element of $T_{(p,\varphi(p))}\Gamma$ has the form (v,φ^*v) for $v\in T_pM$. Then we have

$$(-\omega,\omega)((v,\varphi_*v),(w,\varphi_*w))=-\omega(v,w)+\omega(\varphi_*v,\varphi_*w)=0 \tag{4}$$

since $\varphi^*\omega=\omega$. Note that the fixed points are in bijection with $\Gamma(\varphi)\cap\Delta$ where $\Delta=\{(x,x)\}$ is the diagonal.

Remark 2. If $L\subset M$ is a Lagrangian submanifold (or more generally an immersion) then ω defines a canonical isomorphism of vector bundles from the normal bundle of L to the cotangent bundle of L:

$$NL=TM|_L/TL\xrightarrow{\simeq}T^*L\ . \tag{5}$$

This gives some restrictions on what sort of Lagrangians are allowed.

Example 5. Let $L\subset\mathbb{R}^4$ be a compact connected Lagrangian. Then $\chi(L)=0$ as follows.

More generally, if L is a compact Lagrangian surface in some compact oriented symplectic 4-manifold $\left(M^4,\omega\right)$, then the self-intersection number is $L\cdot L=-\chi(L)$. This is because the isomorphism $\omega\colon NL\to T^*L$ is actually orientation reversing,[2] the Euler number of NL is $L\cdot L$, and the Euler number of T^*L is $\chi(L)$.

Example 6. For $M=\mathbb{R}^4$ and L the x_1,x_2 plane, then $NL=y_1,y_2$ plane.

Example 7. If $L\subset\mathbb{R}^4$ is a compact connected Lagrangian then since $\chi(L)=0$ either $L\simeq T^2$ or $L\simeq$ the Klein bottle. The latter of which is not possible, which was only proved a few years ago.

[1] Professor Hutchings says that if you're ever at a party, and someone asks you what's new in the math world, you should say that they found a new number between 66 and 67 and they're still trying to figure out what's going on with it.

[2] Professor Hutchings says that he has probably spent half of his career figuring out orientation and signs.

Example 8. For $M = \left(S^2 \times S^2, (-\omega, \omega)\right)$, the diagonal Δ is a Lagrangian, and also a sphere. The class is

$$[\Delta] = (1,1) \in H_2(M) = H_2\left(S^2\right) \oplus H_2\left(S^2\right) \ . \tag{6}$$

Generally we have

$$(a_1, b_1) \cdot (a_2, b_2) = -\left(a_1 b_2 + b_1 a_2\right) \ , \tag{7}$$

so in this case we have

$$[\Delta] \cdot [\Delta] = (1,1) \cdot (1,1) = -2 \ . \tag{8}$$

Remark 3. Let X^n be a compact oriented smooth manifold, $A \in H_i(X)$, and $B \in H_j(X)$, where $i + j = n$. In this case we have $A \cdot B \in \mathbb{Z}$. Then the Poincaré duals $A^* \in H^{n-i}(X; \mathbb{Z})$ and $B^* \in H^{n-j}(X; \mathbb{Z})$ satisfy

$$A \cdot B = \langle A^* \smile B^*, [X] \rangle \ . \tag{9}$$

3. A theorem and some proofs

Theorem 2 (Weinstein's Lagrangian Tubular Neighborhood Theorem (WLTN))**.** *If $L \subset (M, \omega)$ is a compact Lagrangian then there is a neighborhood of $L \subset U \subset M$, a neighborhood $U' \subset T^*L$ of the zero-section, and a symplectomorphism $\varphi\colon (U, \omega) \to (U', d\lambda)$ which identifies L with the zero-section in a canonical way.*

Now we prove Darboux's theorem and WLTN theorem. We will use Moser's trick to prove these.

Proposition 3. *Let M be a compact $2n$ dimensional manifold. Let $\{\omega_t\}_{t \in [0,1]}$ be a smooth 1-parameter family of symplectic forms on M. Then the class $[\omega_t] \in H^2(M; R)$ is constant (so independent of t) iff there exists an isotopy $\{\varphi_t\}_{t\in[0,1]}$ (so $\varphi_t \in \mathrm{Diff}(M)$) with $\varphi_0 = \mathrm{id}_M$ and $\varphi_t^* \omega_t = \omega_0$.*

Proof. ($\Longleftarrow$): By homotopy invariance of de Rham cohomology, $\omega_t = \left(\varphi_t^{-1}\right)^* \omega_0$.

($\Longrightarrow$): Consider an isotopy $\{\varphi_t\}_{t\in[0,1]}$ generated by a 1-parameter family of vector fields $\{X_t\}$, i.e. $\varphi_0 = \mathrm{id}_M$ and

$$\frac{d}{dt}\varphi_t(x) = X_t\left(\varphi_t(x)\right) \ . \tag{10}$$

So we know the result is true for $t = 0$, and all we need to show is that

$$\frac{d}{dt}\varphi_t^* \omega_t = 0 \ . \tag{11}$$

We have that

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) \tag{12}$$

$$= \varphi_t^*\left(d\iota_{X_t}\omega_t + \cancel{\iota_{X_t} d\omega_t} + \frac{d}{dt}\omega_t\right) \tag{13}$$

where $\mathcal{L}$ is the Lie derivative, and this term goes to zero since ω_t is closed. Since φ_t is a diffeomorphism, what we want is equivalent to

$$\boxed{d\iota_{X_t}\omega_t = \frac{-d}{dt}\omega_t} \ . \tag{14}$$

Now we want to find X_t solving this equation. We know that $\frac{d}{dt}\omega_t$ is exact. So there exists a 1-form α_t with $d\alpha_t = \frac{d}{dt}\omega_t$. Now we have a technical claim which we will prove later:

Claim 2. We can choose the α_t to depend smoothly on t.

Then with this claim, we just need $d\iota_{X_t}\omega_t = -d\alpha_t$, and to show this, it is enough to get $\iota_{X_t}\omega_t = -\alpha_t$. But now there is a unique X_t satisfying this equation because ω_t is nondegenerate. The vector field X_t depends smoothly on t because ω_t and α_t do. $\square$