LECTURE 6: CONTACT STRUCTURES
MATH 242

LECTURE: PROFESSOR MICHAEL HUTCHINGS
NOTES: JACKSON VAN DYKE

1. Definitions

Contact geometry is meant to be an odd-dimensional version of symplectic geometry. Let Y be a $2n-1$ dimensional smooth manifold.

Definition 1. A *contact form* on Y is a 1-form λ on Y such that $\lambda \wedge d\lambda^{n-1} \neq 0$.

The idea is that some version of the top dimensional exterior product of the form is nonzero. This is the odd-dimensional version of the non-degeneracy of a symplectic form. Having such a form determines two additional things. First, define $\xi = \ker \lambda \subset TY$. This is a codimension 1 sub-bundle.[1] Note that $d\lambda^{n-1}\big|_\xi \neq 0$. Assuming $\lambda \neq 0$ this is equivalent to the non-degeneracy in the definition. This means $d\lambda$ defines a linear symplectic form on ξ_p for each $p \in Y$. In particular, ξ is oriented by $(d\lambda)^{n-1}$.

Definition 2. A contact structure is an oriented co-dimension 1 subbundle of TY such that there exists a contact form λ on Y with $\ker \lambda = \xi$, and $(d\lambda)^{n-1}$ agrees with the orientation on ξ.

By definition any contact form gives rise to a contact structure.

Remark 1. Let λ_1 be a contact form. Then λ_2 is a contact form which gives rise to the same contact structure iff there exists $f : Y \to \mathbb{R} \setminus \{0\}$ (need $f > 0$ if $n-1$ is odd) such that $\lambda_2 = f\lambda_1$.

Proof. ($\Longrightarrow$): By definition, $\ker \lambda_1 = \ker \lambda_2$, so $\lambda_2 = f\lambda_1$ for some non-vanishing function $f : Y \to \mathbb{R} \setminus \{0\}$. The orientation condition forces $f > 0$ for $n-1$ odd.

($\Longleftarrow$): It certainly has the same kernel, so we just need to check that $\lambda_2 = f\lambda_1$ is a contact form. We can calculate:

$$\lambda_2 \wedge (d\lambda_2)^{n-1} = f\lambda_1 \wedge (df \wedge \lambda_1 + f\, d\lambda_1)^{n-1} = f^n \lambda_1 \wedge (d\lambda_1)^{n-1} \neq 0$$

which is never zero because λ_1 is a contact form and f is nonzero. □

Remark 2. Recall that a codimension 1 distribution ξ is integrable if it is a foliation, i.e. locally you can choose coordinates $x_1, \cdots, x_{2n-1}$ such that

$$\xi = \operatorname{Span}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{2n-1}}\right)$$

Date: February 7, 2019.

[1] This is sometimes called a distribution.

Theorem 1 (Frobenius)**.** *A distribution ξ (of any dimension) is a foliation iff we have that if V and W are vector fields with $V(p), W(p) \in \xi_p$ for each $p \in Y$, then the Lie bracket $[V, W](p) \in \xi_p$ for each $p \in Y$.*

Lemma 1. *If λ is a non-vanishing 1-form with $\xi = \ker \lambda$, then ξ is integrable iff $d\lambda|_\xi = 0$.*

Proof. Let V and W be vector fields with $V(p), W(p) \in \xi_p$ for all p. Then we have the formula that:

$$d\lambda(V, W) = V\lambda(W) - W\lambda(V) - \lambda([V, W]) .$$

The first terms are 0 since we assumed V and W to be in $\ker \lambda$. And now we are done by Frobenius' theorem. □

A contact structure is what is called "maximally non-integrable" since $d\lambda|_\xi$ is nondegenerate.

Definition 3. If λ is a contact form, then the *Reeb vector field* R is the unique vector field such that $d\lambda(R, -) = 0$ and $\lambda(R) = 1$.

This is unique because $\dim \ker d\lambda = 1$, and by the non-degeneracy in the definition, λ is not identically zero on the kernel of $d\lambda$.

Warning 1. Multiplying λ by a positive function changes the Reeb vector field. Not just the normalization, the direction too. We should expect this, because if something is in the kernel of $d\lambda$, we shouldn't necessarily expect it to be in the kernel of $df\lambda = df \wedge \lambda + f \wedge d\lambda$.

2. Examples and theorems

Example 1. Take[2] $Y = \mathbb{R}^3$, $\lambda = dz - y\,dx$. Note that $d\lambda = dx\ dy$ which means $\lambda \wedge d\lambda = dx\ dy\ dz$ which is nondegenerate. Then

$$\xi = \ker \lambda = \operatorname{Span}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right) .$$

and $R = \partial/\partial z$. These planes look as in fig. 1.

Remark 3. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a curve which projects to a loop $\bar{\gamma}$ in $\mathbb{R}^2_{x,y}$. Suppose that $\gamma'(t) \in \xi$ for all t. We can consider the difference of the z coordinates:

$$z(\gamma(b)) - z(\gamma(a)) = \int_\gamma dz = \int_\gamma y\,dx = \int_{\bar{\gamma}} y\,dx = -\operatorname{Area}(\bar{\gamma})$$

so a simple closed curve can never lift to a curve tangent to the contact planes. In particular, the obstruction is exactly measured by the signed area of the curve. So if we had a figure 8 with the same amount of area on both sides of the crossing we could somehow close the curve off in $\mathbb{R}^3$ while still staying tangent to ξ. We can visualize this in fig. 1

Definition 4. If Y^3 is a 3-manifold with a contact structure ξ, a *Legendrian knot* in (Y, ξ) is a knot $L \subset Y$ such that $TL \subset \xi$.

[2] Note that some people instead take $\lambda = dz + x\,dy$. Professor Hutchings says such people are heretics.

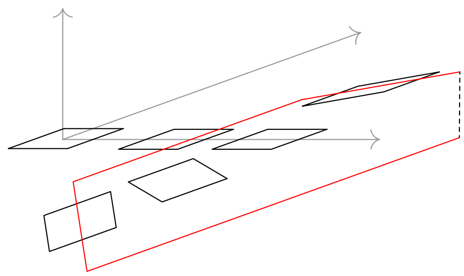

FIGURE 1. Here we show ξ for $\mathbb{R}^3$ where $\lambda = dz + x\,dy$. In red we show the sense in which simple closed curves in the plane fail to lift to to curves in $\mathbb{R}^3$ which are everywhere tangent to the contact planes.

A closed curve $\bar{\gamma}$ in the x, y plane lifts to a Legendrian knot in $\left(\mathbb{R}^3, dz - y\,dx\right)$ iff Area $(\bar{\gamma}) = 0$ and some additional area conditions from the crossings.[3]

Remark 4. Projecting to the x, y plane and considering its area is different from the front projection, which would instead be projecting the knot to the x, z plane so the slope of the projection somehow tells us the y coordinate.

Question 1. Classify Legendrian knots in $\left(\mathbb{R}^3, \lambda_{\text{std}}\right)$ up to Legendrian isotopy. In particular, how different are Legendrian knots from smooth knots?

Note that there is an obvious map from Legendrian knots modulo Legendrian isotopy to smooth knots modulo smooth isotopy which is surjective. It is however not injective. There are some simple invariants which can show certain Legendrian knots are not Legendrian isotopic. The first of which is the rotation number. Consider for example the figure eight presentation of the unknot. This has rotation number 0. If we add an extra turn, the image under the map to smooth knots is still the unknot, however it has rotation number 1, so these are not related by Legendrian isotopy. There is also the Thurston-Bennequin invariant. It was once thought that if this and the rotation number both agreed, then two Legendrian knots were Legendrian isotopic. It was however discovered using Legendrian contact homology that this is not the case.

Example 2. Note that there is a standard contact form on $\mathbb{R}^{2n-1}$ given by

$$\lambda_{\text{std}} = dz - \sum_{i=1}^{n-1} y_i\,dx_i$$

where the coordinates are given by $x_1, \cdots, x_{n-1}, y_1, \cdots, y_{n-1}, z$. Then

$$d\lambda_{\text{std}} = \sum_{i=1}^{n-1} dx_i\ dy_i$$

which means

$$\lambda_{\text{std}} \wedge \left(d\lambda_{\text{std}}\right)^{n-1} = (n-1)!\,dx_1\ dy_1 \cdots dx_{n-1}\ dy_{n-1}\ dz$$

[3]The idea is that if we take one half of the curve split at a crossing then it has to have nonzero signed area.

and again $R = \partial / \partial z$.

The following example should motivate one to care about contact structures.

Example 3. Define a 1-form λ_0 on $\mathbb{R}^{2n}$ by

$$\lambda_0 = \frac{1}{2} \sum_{i=1}^{n} \left(x_i \, dy_i \, - y_i \, dx_i \, \right) \ .$$

Then the claim is as follows.

Claim 1. Let Y be a star-shaped hypersurface in $\mathbb{R}^{2n}$, i.e. it is transverse to the radial vector field

$$\rho = \frac{1}{2} \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) \ .$$

Then we have both of the following:

(1) $\lambda = \lambda_0|_Y$ is a contact form on Y.
(2) R is proportional to X_H for any Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ having Y as a regular level set.

Proof. Observe that $\lambda_0 = \iota_\rho \omega_{\text{std}}$. So we want to show that $\lambda = \iota_\rho \omega_{\text{std}}|_Y$ is a contact form. First note that $\lambda|_Y \neq 0$ because $\rho \pitchfork Y$. [$\rho(y) \in \ker(\iota_\rho \omega_{\text{std}}(y))$, and if $T_y Y \subset \ker(\iota_\rho \omega_{\text{std}}(y))$ then $\ker(\iota_\rho \omega_{\text{std}}) = T_y \mathbb{R}^{2n}$ which is a contradiction since $\rho \neq 0$, and ω_{std} is nondegenerate.] To show that λ is a contact form, we need to check that $\left(d\lambda \right)^{n-1}\Big|_{\ker \lambda}$ is nondegenerate. This follows because $d\lambda = \omega_{\text{std}}|_Y$ and $\left(\omega_{\text{std}}|_Y \right)^{n-1} \neq 0$.

The following is the more general form of this fact:

Theorem 2. *In a symplectic manifold (M, ω), let Y be a hypersurface. Let ρ be a vector field in a neighborhood of Y such that $\rho \pitchfork Y$ and $d\iota_\rho \omega = \omega$. (Note this is what is called a Liouville vector field.) Then $\lambda = (\iota_\rho \omega)|_Y$ is a contact form on Y.*

Recall that if $y \in Y$, then $\omega|_{T_y Y}$ has a 1-dimensional kernel, call it L_y, and $X_H(y)$ is a generator of L_y. Observe that $R \in L_y$ because of the following. We know $d\lambda(R, -) = 0$, but $d\lambda = \omega|_Y$, so both the Hamiltonian vector field and the Reeb vector field have to be in the same one-dimensional space. □