LECTURE 7
MATH 242

LECTURE: PROFESSOR MICHAEL HUTCHINGS
NOTES: JACKSON VAN DYKE

1. Contact manifolds

Recall a *contact form* λ is a 1-form such that $\lambda \wedge (d\lambda)^{n-1} \neq 0$. This gives rise to the contact structure $\xi = \ker \lambda$ and a Reeb vector field R which is characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.

1.1. Why we care.

Let Y be a compact smooth hypersurface in a symplectic manifold (M^{2n}, ω).

Definition 1. *Y is contact type if there exists a contact form λ on Y such that $d\lambda = \omega|_Y$.*

Note that any hypersurface Y in (M, ω) has a characteristic foliation L which is a rank 1 foliation with $L_y = \ker\left(\omega|_{T_y Y}\right)$. If $H : M \to \mathbb{R}$ is a Hamiltonian, with Y as a regular level set, then for $y \in Y$ we have $X_H(y) \in L_y$.

Remark 1. If Y is contact-type then the Reeb vector field R has the property that $R(y) \in L_y$.

Note that this implies that periodic orbits of R correspond to periodic orbits of X_H. So if you're interested in one, you're interested in the other.

Conjecture 1 (Weinstein). *If Y^{2n-1} is a closed manifold, then every contact form on Y has a Reeb orbit.*

In the 70s, Rabinowitz proved this for star-shaped hypersurfaces in $\mathbb{R}^{2n}$ In the 80s, Viterbo proved this for contact-type compact hypersurfaces in $\mathbb{R}^{2n}$. In 2006, Taubes proved this for all closed 3-manifolds. This is an open question for higher dimensions.

2. Liouville vector fields

Recall that if (M, ω) is symplectic, a Liouville vector field on M is a vector field X such that $\mathcal{L}_X \omega = d\iota_X \omega = \omega$.

Example 1. Let $M = \mathbb{R}^{2n}$ with the standard form

$$\omega = \sum_{i=1}^{n} dx_i \, dy_i$$

Date: February 12, 2019.

then the Liouville vector field is just the radial one:

$$X = \frac{1}{2}\sum_{i=1}^{n}\left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right) .$$

Indeed,

$$\iota_X \omega = \frac{1}{2}\sum_{i=1}^{n}\left(x_i \, dy_i - y_i \, dx_i\right)$$

which clearly satisfies $d\iota_X\omega = \omega$.

Lemma 1. *A compact hypersurface $Y \subset (M, \omega)$ is contact type iff there exists a Liouville vector field X in a neighborhood of Y with $X \pitchfork Y$ (e.g. a star-shaped hypersurface in $\mathbb{R}^{2n}$).*

Proof. ($\Longleftarrow$): Define $\lambda = \iota_X \omega|_Y$. The definition implies that $d\lambda = \omega|_Y$. We need to check that λ is a contact form. Let $y \in Y$ and $v \in L_y \setminus \{0\}$. Then $\omega(V, X) \neq 0$ In particular, $\omega(V, X) = -\lambda(V)$. Then $\ker \lambda|_{T_y Y}$ is the symplectic complement of $\operatorname{Span}(V, X)$. This is because if $w \in \ker\left(\lambda|_{T_y Y}\right)$ then

$$\omega(W, X) = -\iota_X \omega(W) = -\lambda(W) = 0 .$$

It follows that $d\lambda$ is nondegenerate on $\ker\left(\lambda|_{T_y Y}\right)$. This, along with the above calculation that $\omega(V, X) = -\lambda(V)$, is enough to see that $\lambda \wedge (d\lambda)^{n-1} \neq 0$. ☐

Before we finish the proof, we need another definition.

Definition 2. Let (Y, λ) be a contact manifold. Define the symplectization to be the symplectic manifold

$$(\mathbb{R}_s \times Y, \omega = d(e^s \lambda)) .$$

We now check this is symplectic. First we have that $d\omega = 0$. Note that

$$\omega = e^s \left(ds \wedge \lambda \, d\lambda\right)$$

so we have that

$$\omega^n = n e^{ns} \, ds \wedge \underbrace{\lambda \wedge (d\lambda)^{n-1}}_{\text{nonzero on } Y} \neq 0 .$$

This is the volume form, so we should think of this as somehow telling us that this thing blows up in the s direction.

Remark 2. $\{s\} \times Y$ is a contact type hypersurface for any value $s \in \mathbb{R}$. The contact form is $e^s \lambda$.

Continued proof of lemma 1. ($\Longrightarrow$): Suppose λ is a contact form on Y with $d\lambda = \omega|_Y$. Then we need to find a transverse Liouville vector field of X in a neighborhood of Y. Choose a section $\hat{X}$ of $TM|_Y$ such that

$$\hat{X} \in (\ker \lambda)^{\omega} \qquad\qquad \omega\left(\hat{X}, R\right) = 1 .$$

Extend $\hat{X}$ arbitrarily to a non-vanishing vector field of Y. Note that $\hat{X} \pitchfork Y$. We can choose this in a neighborhood to be identified with $(-1, 1)_s \times Y$ such that $Y \leftrightarrow \{0\} \times Y$.

On this neighborhood we have a symplectic form $\hat{\omega} = d(e^s \lambda)$ and $\hat{X} = \partial / \partial s$ is a Liouville vector field for $\hat{\omega}$. Note that the point here is that along Y, $\hat{\omega} = \omega$.

Recall that the relative Moser theorem implies that after shrinking neighborhoods we get a diffeomorphism φ from a neighborhood of Y in M to a neighborhood of $\{0\} \times Y$ in $(-1,1) \times Y$ such that $\varphi|_Y = \mathrm{id}|_Y$ and $\varphi^*\hat{\omega} = \omega$. Then $\varphi^*\hat{X}$ is a Liouville vector field for ω which is transverse to Y. $\square$

Remark 3. There exist compact hypersurfaces in $\mathbb{R}^{2n}$ for $n > 1$ (C^2 smooth if $n = 2$, C^∞ if $n > 2$) such that X_H (or L) has no periodic orbit.

It follows from Viterbo's theorem that these are not contact type.

Question 1. Is every embedded S^3 in $\mathbb{R}^4$ isotopic to a contact type hypersurface?

This would be a big deal to answer, since it would imply the Schoenflies conjecture, which says that every embedded S^3 in $\mathbb{R}^4$ bounds a ball.

3. Examples of contact manifolds

Example 2. Let Z be a smooth manifold with a Riemannian metric g. Take

$$T = ST^*Z = \left\{(y,v) \in T^*Y \mid \|v\|_g = 1\right\} ,$$

the unit cotangent bundle. Recall that T^*Z is symplectic where

$$\lambda = \sum_{i=1}^n p_i \, dq_i$$

in local coordinates and $\omega = d\lambda$. The Liouville vector field is

$$X = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i} \ .$$

So $X \pitchfork Y$, and therefore $\lambda|_Y$ is a contact form, and R is the geodesic flow.

Example 3. More generally, if $Y \subset T^*Z$ is a hypersurface which is "fiberwise star-shaped", i.e. transverse to

$$X = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i} \ ,$$

then Y is contact type.

Example 4. Let

$$Y = T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$$

with coordinates x, y, z. Take

$$\lambda = (\cos z)\ dx + (\sin z)\ dy \ .$$

We check this is a contact form by calculating

$$d\lambda = -\sin z \, dz\ dx + \cos z \, dz\ dy$$

and

$$\lambda \wedge d\lambda = \cos^2\ dx\ dz\ dy - \sin^2 z \, dy\ dz\ dx = -\, dx\ dy\ dz \ .$$

One might think this should be positive, but it isn't.[1] The Reeb flow is:

$$R = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y} \ .$$

[1]Professor Hutchings offers a cautionary tale about signs: there was once a theorem relating the cohomology of cotangent bundles to loop spaces. There were three independent proofs of this, but they were all wrong because of a sign.

We could get a variant of this example by setting:

$$\lambda = (\cos nz)\ dx\ + (\sin nz)\ dy\ .$$

Fact 1. *The contact structures for different n are not contactomorphic.*

Example 5 (Boothby-Wang manifolds/pre-quantization spaces)**.** This is a way to start with a symplectic manifold, and create a contact manifold with one higher dimension. Let $\left(M^{2n}, \omega\right)$ be a closed symplectic manifold. Suppose $-[\omega]/(2\pi) \in H^2(M;\mathbb{R})$ is the image of an integral class $e \in H^2(M;\mathbb{Z})$. Let Y^{2n+1} be the principle S^1-bundle over M with Euler class e.

First we review these concepts. Let B be any smooth manifold. Recall a principal S^1-bundle over B consists of

- a smooth manifold E
- a map $\rho : E \to B$
- an S^1 action on E

such that:

- ρ is surjective
- S^1 preserves the fibers of ρ
- S^1 acts freely and transitively on each fiber,
- for each point in B, we can find a neighborhood $U \subseteq B$ such that

$$\begin{array}{ccc} \rho^{-1}(U) & \simeq & U \times S^1 \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & U & \end{array} .$$

Remark 4. We could also define a principle G bundle for any Lie group G, but this gets a bit tricky to deal with when G is nonabelian, so we won't worry about this for now.

So consider such a bundle $\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$.

Definition 3. A connection on a principal S^1-bundle is a 1-form A on E such that

- A is invariant under the S^1 action
- $A(V) = 1$ where V is the derivative of the S^1 action.

The point of defining this, is that a connection is equivalent to an S^1-invariant splitting of the short exact sequence

$$0 \longrightarrow T_e\rho^{-1}(b) \underset{\overset{A}{\dashleftarrow}}{\longrightarrow} T_eE \overset{\rho_*}{\underset{\text{horizontal lift}}{\rightleftarrows}} T_bB \longrightarrow 0$$

for $b \in B$, $e \in \rho^{-1}(b)$. We call the first term the vertical subspace. We want to somehow consider a horizontal subspace, but there is no canonical way to make this choice, so such a splitting is somehow a choice of such a subspace. This is equivalent to the connection A.

Then the point here will be that an appropriate connection is a contact form.