LECTURE 8
MATH 242

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1. S^1-bundles

Let B be a CW-complex.

Definition 1. An S^1 bundle over B is a continuous surjective map $\pi : E \to B$ such that for each $p \in B$ there is a neighborhood U of p such that

$$\begin{tikzcd} \rho^{-1}(U) \arrow[dr, "\pi"'] & \simeq & U \times S^1 \arrow[dl] \\ & U & \end{tikzcd} .$$

Remark 1. E is trivial, i.e. $E = B \times S^1$, iff it has a section $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

Definition 2. An *oriented* S^1*-bundle* is an S^1-bundle in which each fiber has an orientation which depends continuously on p.

1.1. Euler class.

The *Euler class* is a cohomology class $e(E) \in H^2(B, \mathbb{Z})$ associated to the bundle E. In particular, this is an obstruction to E being trivial, i.e. an obstruction to finding a section.

To see this, we will attempt to find a section $s : B \to E$ one cell at a time. Over the 0-skeleton we have no problem, since $S^1 \neq \emptyset$. The bundle over the 1-cell is no problem because S^1 is connected. In particular, the 1-cell can be identified with an interval, and since S^1 is connected, we get a path living over it. Now we try to extend it over the 2-skeleton. For a 2-cell $\sigma : D^2 \to B$ we have $\sigma^* E \simeq D^2 \times S^1$. And now we have an obstruction $\mathfrak{o}(\sigma) \in \pi_1(S^1) = \mathbb{Z}$. We can think of $\mathfrak{o} \in C^2(B, \mathbb{Z})$.

Lemma 1.
- $\delta \mathfrak{o} = 0$
- *If $\mathfrak{o}'$ comes from a different choice of section over the* 1*-skeleton, then $\mathfrak{o}' - \mathfrak{o} = \delta(\text{something})$.*

The conclusion is that we have a well-defined cohomology class in $H^2(B; \mathbb{Z})$.

So from the first item we have that if we have a 3-cell, the sum of the obstructions for the boundary 2-cells is 0. By construction, if $e(E) = 0$ then E certainly has a section over the 2-skeleton. As it turns out, this is also equivalent to E having a section since $\pi_k(S^1) = 0$ for $k > 1$.

We can also show that the Euler class, viewed as a map

$$e : \frac{\left\{\text{oriented } S^1\text{-bundles over } B\right\}}{\text{isomorphism}} \xrightarrow{\simeq} H^2(B, \mathbb{Z}) ,$$

is an isomorphism.

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1.2. **Principal S^1-bundles.** Recall a principal S^1 bundle $\begin{array}{c} E \\ \downarrow \rho \\ B \end{array}$ is an S^1-bundle such that S^1 acts freely and transitively on the fibers. Write V for the derivative of the S^1 action, i.e. a vector field on E tangent to the fibers.

Also recall that a *connection* on E is a real-valued 1-form A on E such that:

- A is invariant under the S^1 action
- $A(V) = 1$

This is equivalent to splitting the SES:

$$\begin{tikzcd} 0 \arrow[r] & T_e\rho^{-1}(b) = \mathbb{R} \arrow[r] & T_eE \arrow[r, "\rho_*"] \arrow[l, dashed, bend right, "A"'] & T_bB \arrow[r] \arrow[l, dashed, bend left, "\text{horizontal lift}"] & 0 \end{tikzcd}$$

for $b \in B$ and $e \in E$.

So let A be a connection. S^1 invariance implies the Lie derivative $\mathcal{L}_V A = 0$. By Cartan's formula this says:

$$0 = \mathcal{L}_V A = d \underbrace{\iota_V A}_{=1} + \iota_v \, dA = \iota_V \, dA \ .$$

This together with S^1-invariance implies that there exists a 2-form ω on B such that $dA = \rho^*\omega$. If $b \in B$ and $X_1, X_2 \in T_bB$, then pick $e \in \rho^{-1}(b)$ and $\tilde{X}_1, \tilde{X}_2 \in T_eE$ with $\rho_*\tilde{X}_1 = X_1$, $\rho_*\tilde{X}_2 = X_2$. Then define

$$\omega(X_1, X_2) = dA\left(\tilde{X}_1, \tilde{X}_2\right) \ .$$

Now we need to check this is well defined. This does not depend on $\tilde{X}_1$ and $\tilde{X}_2$ since $\iota_V \, dA = 0$. This does not depend on E because dA is S^1-invariant. This is known as the *curvature* of the connection A.

For the trivial bundle $B \times S^1$, we can pull A back from S^1 to get what is called the trivial connection.

Observe that $d\omega = 0$. This is because $dA = \rho^*\omega$ and

$$0 = ddA = d\rho^*\omega = \rho^* \, d\omega$$

which implies $d\omega = 0$ since ρ is surjective on tangent vectors. This means that ω defines a cohomology class $[\omega] \in H^2(B, \mathbb{R})$.

Lemma 2. *This cohomology class does not depend on the choice of connection.*

The punchline is the following theorem:

Theorem 1.
$$-\frac{[\omega]}{2\pi} \in H^2(B; \mathbb{R})$$
is the image of the Euler class $e \in H^2(B; \mathbb{Z})$ under the map $H^2(B, \mathbb{Z}) \to H^2(B, \mathbb{R})$.

In short:

$$[\omega] = -2\pi e \ .$$

Proof (for the case of B a surface). Assume B is a compact, connected, and oriented surface. Let $p \in B$ be the center of a 2-cell D in a triangulation. We want to think of D as a small disk. We can find a section $s : B \setminus \operatorname{int} D \to E$

where $E|_D \simeq D \times S^1$ and $s|_{\partial D} : \partial D = S^1 \to S^1$ which represents the class $e(E) \in \pi_1(S^1) = \mathbb{Z}$. Identifying $H^2(B;\mathbb{R}) = \mathbb{R}$ we have

$$[\omega] = \int_B \omega \approx \int_{B \setminus \operatorname{int} D} \omega \ .$$

Recall $dA = \rho^* \omega$. For a section s,

$$s^* \, dA = s^* \rho^* \omega = (\rho \circ s)^* \omega = \omega$$

so

$$\int_{B \setminus \operatorname{int} D} \omega = \int_{B \setminus \operatorname{int} D} s^* \, dA = \int_{B \setminus \operatorname{int} D} ds^* A = -\int_{\partial D} s^* A$$

by Stokes' theorem. Then we get that

$$-\int_{\partial D} S^* A \approx -2\pi \operatorname{wind}(S|_{\partial D}) = -2\pi e$$

and in the limit as the disk goes to 0, the approximations cancel. $\square$

2. Contact manifolds

Let (M^{2n}, ω) be a compact symplectic manifold. Suppose there exists $e \in H^2(X;\mathbb{Z})$ such that $[\omega] = -2\pi e$. Let $\begin{array}{c} E \\ \downarrow \rho \\ M \end{array}$ be a principal S^1-bundle over M with Euler class e. Let A be a connection 1-form with curvature ω. We can find such things by the following exercises:

Exercise 1. Show we can always construct an S^1-bundle with a given Euler class.

Exercise 2. Consider a connection A with curvature ω. Then prove we can adapt A such that the corresponding curvature is any other 2-form in the same cohomology class of ω.

Then A is a contact form on E. To see this we need to check that

$$A \wedge (dA)^n = A \wedge \rho^* \omega^n$$

is nonzero, but this follows from the fact that A is nonzero on V, and ω^n is nonzero on B. Then the Reeb vector field is just $R = V$. This is because by definition we have $dA(R, \cdot) = 0$ and $A(V) = 1$ as desired. Therefore every fiber is a Reeb orbit. These examples are called pre-quantization spaces or Boothby-Wang manifolds.

Remark 2. If E is an S^1 bundle over a surface B, this is very different from the canonical contact form on the unit cotangent bundle ST^*B.

2.1. **Some lemmas.**

Lemma 3 (Darboux for contact forms)**.** *For any contact form λ on Y^{2n+1} and $p \in Y$, there are local coordinates around p in which*

(1)
$$\lambda = dz - \sum_{i=1}^{n} y_i \, dx_i \ .$$

Proof. By some linear algebra, one can choose local coordinates such that (1) holds at p. Then let λ_0 be the standard contact form in these coordinates, and $\lambda_1 = \lambda$. So we want to pull λ_1 back to λ_0.

Let $\lambda_t = (1-t)\lambda_0 + t\lambda_1$. Note this is a contact form near p. Now we want to do the Moser trick again. We want a family of diffeomorphisms φ_t from some neighborhood of p to some neighborhood of p, with $\varphi_t^*\lambda_t = \lambda_0$ for $t \in [0,1]$, $\varphi_0 = \mathrm{id}$, and $\varphi_t(p) = p$ for all t. So we want to solve for a vector field X_t which corresponds to the $\{\varphi_t\}$. So we need

$$0 = \frac{d}{dt}\varphi_t^*\lambda_t = \varphi_t^*\left(\frac{d\lambda_t}{dt} + \mathcal{L}_{X_t}\lambda_t\right) = \varphi_t^*\left(\frac{d\lambda_t}{dt} + d\iota_{X_t}\lambda_t + \iota_{X_t}d\lambda_t\right) .$$

We know that $X_t = f_t R_t + Y_t$ where R_t is the Reeb vector field for λ_t, for some f_t and Y_t which is in $\xi_t = \ker(\lambda_t)$. So we just need to solve for these. We need

$$\frac{d\lambda_t}{dt} + df_t + \iota_{Y_t}d\lambda_t = 0$$

and now we can plug in R_t to get

$$\frac{d\lambda_t}{dt}R_t + R_t f_t = 0$$

and now we can solve for f_t as long as our neighborhood is small enough that R_t has no closed orbits. Then we can solve for λ_t by non-degeneracy of $d\lambda_t$. $\square$