LECTURE 9
MATH 242

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1. More contact structures

Lemma 1. *Let Y be a compact $2n-1$ dimensional manifold. Let $\{\xi_t\}_{t\in[0,1]}$ be a smooth family of contact structures., i.e. $\xi_t = \ker(\lambda_t)$ where $\{\lambda_t\}$ is a smooth family of contact forms. Then there exists an isotopy $\{\varphi_t\}_{t\in[0,1]}$, for $\varphi_t \in \mathrm{Diff}(Y)$, $\varphi_0 = \mathrm{id}_R$ such that $\varphi_t^*\xi_t = \xi_0$ for each t, i.e. $\varphi_t^*\lambda_t = f_t\lambda_t$ for smooth $f_t : Y \to \mathbb{R}^{\geq 0}$.*

Remark 1. We cannot expect to get an isotopy $\varphi_t^*\lambda_t = \lambda_0$, because this would imply that the dynamics would be the same under φ_t, but this can't be the case. E.g. recall that $\partial E(a,b)$ (with the restriction of the standard form on $\mathbb{R}^4$) has simple Reeb orbits of period a, b, and no others if $a/b \notin \mathbb{Q}$. If there exists a diffeomorphism with $\varphi^*\lambda_{a,b} = \lambda_{a',b'}$ then the Reeb orbits for $\lambda_{a,b}$ have the same period as the Reeb orbits for $\lambda_{a',b'}$.

Proof. We want to find φ_t and f_t with

$$\varphi_t^*\lambda_t = f_t\lambda_0 \ .$$

Now take the derivative of both sides to get:

$$\frac{d}{dt}\varphi_t^*\lambda_t = \left(\frac{d}{dt}f_t\right)\lambda_0$$

but the LHS can be written:

$$\frac{d}{dt}\varphi_t^*\lambda_t = \varphi_t^*\left(\frac{d\lambda_t}{dt} + \mathcal{L}_{X_t}\lambda_t\right) = \left(\frac{d}{dt}f_t\right)\left(\frac{1}{f_t}\varphi_t^*\lambda_t\right)$$

and now we need to solve for this X_t. Rewrite this as

$$\varphi_t^*\left(\frac{d\lambda_t}{dt} + \mathcal{L}_{X_t}\lambda_t\right) = \varphi_t^*(g_t\lambda_t) \qquad \text{where} \qquad \varphi_t^* g_t = \frac{1}{f_t}\frac{df_t}{dt} \ .$$

So we just need to solve:

$$\frac{d\lambda_t}{dt} + \mathcal{L}_{X_t}\lambda_t = g_t\lambda_t \ .$$

First we can rewrite the Lie derivative to get:

$$\frac{d\lambda_t}{dt} + d\iota_{X_t}\lambda_t + \iota_{X_t}d\lambda_t = g_t\lambda_t$$

Let us take $X_t(p) \in \xi_t(p)$ for each p, so $\lambda_t(X_t) = 0$, so the second term is 0. Now we can uniquely decompose

$$\frac{d\lambda_t}{dt} = g_t\lambda_t + h_t$$

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where h_t annihilates R_t, i.e. $g_t = d\lambda/dt\,(R_t)$. We need $\iota_{X_t}\, d\lambda_t = h_t$. But we can uniquely solve for X_t by non-degeneracy of $d\lambda_t$ on ξ_t. Now we can solve

$$\varphi_t^* g_t = \frac{d}{dt}\log(f_t)$$

to get f_t. □

1.1. **Contact structures on 3-manifolds.** Let ξ be a contact structure on Y^3.

Definition 1. ξ is called *overtwisted* if there exists an embedded disk $D \subset Y$ such that $\xi|_{\partial D} = TD|_{\partial D}$. Such a disk is called an *overtwisted disk*.

Example 1. On $\mathbb{R}^3$ with cylindrical coordinates r, θ, z, take

$$\lambda = \cos r\, dz + r \sin r\, d\theta \ .$$

Note

$$d\theta = \frac{x\, dy - y\, dx}{r^2}$$

but $r \sin r = r^2$ (some smooth function) so this is indeed a well-define 1-form. Then we calculate:

$$d\lambda = -\sin r\, dr\ dz + (\sin r + r\cos r)\ dr\ d\theta$$

and

$$\begin{aligned} \lambda \wedge d\lambda &= \cos r\,(\sin r + r\cos r)\ dz\ dr\ d\theta - r\sin^2 r\, d\theta\ dr\ dz \\ &= (r + \cos r \sin r)\ dz\ dr\ d\theta \end{aligned}$$

so this is a fine contact form.

Now the contact structure is:

$$\xi = \operatorname{Span}\left(\frac{\partial}{\partial r}, r\sin r\frac{\partial}{\partial\theta} - \cos r\frac{\partial}{\partial z}\right) \ .$$

Then this will be flat at the origin, and on the locus where $z = 0$ and $r = \pi$. In particular, the disk of radius π is an over-twisted disk.

Note that if we took the standard contact form instead of this weird one, we wouldn't have gotten this behaviour.

Definition 2. ξ is tight if it is not over-twisted.

Fact 1. *The standard contact structure* $\xi = \ker(dz - y\, dx)$ *on* $\mathbb{R}^3$ *is tight.*

This was first proved by Bennequin in the early 80s. Nowadays it follows from holomorphic curve techniques. In particular, it is a special case of the following theorem.

Definition 3. A *strong symplectic filling* of a compact manifold (Y^{2n-1}, ξ) with a contact structure, is a compact symplectic manifold (X^{2n}, ω) with boundary Y such that either

(1) there exists a contact form λ on Y with $\ker\lambda = \xi$, and $d\lambda = \omega|_Y$. Note that when we write $\partial X = Y$, this is with orientation[1],
(2) or equivalently there exists a Liouville vector field V on X in a neighborhood of Y such that $V \pitchfork Y$, $\ker \iota_V\, \omega|_Y = \xi$, and V points <u>out</u> of Y.

Theorem 1. *If* (Y^3, ξ) *has a strong symplectic filling, then* ξ *is tight.*

[1] Both X and Y have orientations induced by their non-degeneracy conditions.

We will (hopefully) prove this later using holomorphic curves.

Example 2. If Y is a compact star-shaped hypersurface in $\mathbb{R}^4$, then this separates $\mathbb{R}^4$ into two regions, and the compact region is a strong symplectic filling of Y.

Example 3. Let U be the unit cotangent bundle of a surface (Σ, g), then a strong filling is given by cotangent vectors of length ≤ 1.

Theorem 2 (Eliashberg). *Fix a compact 3-manifold. The inclusion map from over-twisted contact structures on Y to oriented 2-plane fields on Y is a homotopy equivalence.*

In particular, every orientable 3-manifold has an oriented 2-plane field.

Fact 2. *There exist compact 3-manifolds with no tight contact structure.*

Theorem 3 (Eliashberg). *Suppose $\left(Y^3, \xi\right)$ is tight and Y is closed. Let $\Sigma \subset Y$ be an embedded, connected, orientable surface. Then*

$$\begin{cases} 2g-2 & \Sigma \neq S^2 \\ 0 & \Sigma = S^2 \end{cases} = \max\left\{0, -\chi\left(\Sigma\right)\right\} \geq \left|\left\langle [\Sigma], e\left(\xi\right)\right\rangle\right|$$

where $[\Sigma] \in H_2(Y)$ and $e(\xi) \in H^2(Y, \mathbb{Z})$.

One thing this gives us is a lower bound on the Thurston norm i.e. the minimum genus needed for a surface to represent the same homology class. This also gives restrictions on $e(\xi)$ for tight ξ.

2. Geodesic flow

Now we want to return to a proof that we did not do before. In particular, we will show that on T^*X, X_H for $H = |\cdot|^2/2$ yields exactly geodesic flow. This comes from the relationship between Hamiltonian mechanics and Lagrangian mechanics.

Let X be a smooth n-dimensional manifold, and let $L : TX \to \mathbb{R}$ be a Lagrangian[2] function. One version of classical mechanics is the following. A trajectory $\gamma : [a, b] \to X$ is a critical point of the "action"

$$A(\gamma) = \int_a^b L(\gamma, \dot{\gamma})\, dt$$

subject to the boundary conditions

$$\gamma(a) = x_0 \qquad \gamma(b) = x_1 \ .$$

Example 4. If X has a Riemannian metric, and $L(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2$, then these critical points are geodesics from x_0 to x_1 parametrized at constant speed.

Now we can work in local coordinates $x_1, \cdots, x_n$ on X and $v_1, \cdots, v_n$ on TX. We can write our path as:

$$\gamma(t) = (x_1(t), \cdots, x_n(t))$$

and then we can consider a variation

$$\eta : [a, b] \to \mathbb{R}^n$$

[2]This is different from a Lagrangian submanifold.

such that $\eta(a) = \eta(b) = 0$ so we don't change our boundary conditions. We can also write $\eta(t) = (\eta_1(t), \cdots, \eta_n(t))$. Then γ is a critical point iff for all such η

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} A(\gamma + \epsilon\eta) = 0 \ .$$

Now let's see what this tells us. We can write

$$\begin{aligned} \frac{d}{d\epsilon} A(\gamma + \epsilon\eta) &= \int_a^b \sum_{i=1}^n \left(\eta_i \frac{\partial L}{\partial x_i} + \frac{\partial \eta_i}{\partial t}\frac{\partial L}{\partial v_i}\right) dt \\ &= \int_a^b \sum_{i=1}^n \eta_i \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i}\right) \end{aligned}$$

where we have integrated by parts. Since this has to work for any η, γ is a critical point iff

$$\frac{\partial L}{\partial x_i}(\gamma, \dot{\gamma}) - \frac{d}{dt}\frac{\partial L}{\partial v_i}(\gamma, \dot{\gamma}) = 0$$

for all i. These are called the *Euler-Lagrange equations.*

So we're starting by looking for a map to the tangent bundle satisfying some equations, and we're trying to get a map to the cotangent bundle to get back to symplectic geometry. We will leave this as a cliffhanger.[3] We will start talking a bit about group actions in symplectic geometry soon, and then move on to pseudo-holomorphic curves. Professor Hutchings will be gone on Tuesday March 5th.

[3]This is in the beginning of McDuff-Salamon.