LECTURE 1
MATH 256

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1. Logistics

There is a course site. The guiding text for the course will be EGA. This is however only in French, so some synopses are posted on the site. In addition to this, we will use Ravi Vakil's notes "the rising sea" which are very much done in the style of EGA, though it actually has examples.

There will be no exams, homework will be posted from time to time. Try to do at least half of it. Office hours are posted on the webpage, but it might be best to make an appointment as these are shared by math 55...

2. Course outline

For the first couple of weeks we will just be doing examples. We won't meet a theorem until we start actually talking about schemes. So what is algebraic geometry? It is the geometry of a particular kind of space. In particular, spaces that can be described "locally" by polynomial[1] equations in coordinates. As it turns out, the right way to formulate this is the theory of *schemes*.

One thing that's tricky to get a handle on, is that classical algebraic geometry is when we're talking about finite complex coordinates or some algebraically closed fields. Schemes are however more general. That is, classical varieties are reduced schemes of finite type over an algebraically closed field such as $\mathbb{C}$.

But why consider such spaces? There are two aspects to this question. On one hand, this is a very rigid sort of geometry, so it's sort of a subset of other sorts of geometry. Complex algebraic varieties are a special case of $\mathbb{C}$-analytic varieties. That is we allow analytic functions instead of only polynomials. We can also consider smooth complex algebraic varieties, which is a special case of a $\mathbb{C}$-manifold. Then all of this is just a special case of topological spaces. Algebraic geometry is somehow the most structured version of geometry. This means it has a deeper and more powerful theory. On the other hand, it is remarkably comprehensive. All sort of interesting spaces are algebraic varieties.

3. Examples

We will do some examples of classical algebraic geometry to get a picture in our heads before we actually define schemes. We consider some classical affine varieties. Take an algebraically closed field $k = \overline{k}$. Our coordinate space will be k^n, then a

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[1]This is what makes it *algebraic*.

variety is a subset $X \subseteq k^n$ defined by polynomial equations with n variables. We can always assume these are of the form $f(\underline{x}) = 0$ for some polynomial f.

$$\mathcal{F} = \{\text{polynomials } f_1, \cdots\} \subseteq k[X_1, \cdots, X_n]$$

Then the variety itself will be

$$X = V(\mathcal{F}) = \{\underline{x} \mid \forall f \in \mathcal{F}, f(\underline{x}) = 0\}$$

Example 1. The empty set $\emptyset$ is the solution set of the equation $1 = 0$. Also, the whole space $k^n = V(\emptyset) = V(\{0\})$ is a variety.

Take any point $(x_1, \cdots, x_n) \in k^n$. This is a variety:

$$X = \{(x_1, \cdots, x_n)\} = V(X_1 - x_1, \cdots, X_n - x_n)$$

Example 2 (Plane conics)**.** Let $n = 2$, then consider the variety $C = V(f) \subseteq k^2$ for some quadratic

$$f(X, Y) = aX^2 + bXY + cY^2 + dX + eY + g$$

such that a,b,c are not all 0. First there are some degenerate cases which occur when f factors. For example, if $f(X, Y) = XY$, this is just the union of the X and Y axis in the plane. If this factored differently, it would still be two lines, but they wouldn't be the axes, in fact they might not even meet, such as $F(X, Y) = X(X - 1)$. These seem like two different cases, but really they are the same, because if we consider this inside projective space $\mathbb{P}^2$, then the parallel lines do meet, they just happen to meet at infinity. In fact, there is an even more degenerate case, when $X^2 = 0$. We should think of this as a line which is "double-thick." This sort of thing is handled well by the theory of schemes. The generic case is when f is irreducible. Consider the following four varieties:

$$V\left(X^2 + Y^2 - 1\right) \quad V\left(X^2 - Y^2 - 1\right) \quad V\left(-X^2 + Y^2 - 1\right) \quad V\left(X^2 - Y\right)$$

Let's consider the $\mathbb{R}$ case we saw in high-school. The first one is a circle, the second is a hyperbola, the third is empty, and the last one is a parabola. So there are sort of three kinds of nondegenerate nonempty conics. But what if we allow $\mathbb{C}$ solutions? If we have a solution to the first equation, we can just multiple y by i to get a solution to the second equation, and then we get a solution to the third by multiplying both coordinates by i. So the first three are sort of the same over $\mathbb{C}$ in the sense that their solutions are isomorphic. The parabola is a bit different, but we do have the following:

Fact 1. *All nondegenerate projective conics in $\mathbb{P}^2$ are the same.*