LECTURE 10
MATH 256A

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The website will eventually be updated with exercises and reading etc.

1. Hypersurfaces

Hypersurfaces are defined by a single homogeneous polynomial. Note that we need it to be of degree two for it to even be interesting, so let $f(x_0, \cdots, x_n)$ be of degree 2, then $H = V(f) \subset \mathbb{P}^n$ is what is called a quadric.

Recall we can always write a homogeneous quadratic form $f(x_0, \cdots, x_n) = \underline{x}^T A \underline{x}$ where A is some symmetric matrix. In particular, up to a linear change of variables, this A is unique i.e. it is only determined up to changes of the following form: $S^T AS$. Then it is a basic result of linear algebra that over an algebraically closed field, we can always get A to be a diagonal matrix with r 1s and potentially some 0s, so we can assume that f is just $x_0^2 + \cdots + x_r^2$ up to isomorphism.

Now for any element of our projective space $(x_0 : \cdots : x_r : x_{r+1} : \cdots : x_n)$ if we insist that $x_0^2 + \cdots + x_r^2 = 0$, this doesn't affect the rest of the coordinates. The picture is as in fig. 1 where

$$Y = V(x_{r+1}, \cdots, x_n) \cong \mathbb{P}^r \qquad Z = V(x_0, \cdots, x_r) \cong \mathbb{P}^{n-r-1}$$

Every point in the projective space lies on one of the projective lines connecting Z to Y.

Now we impose an equation inside Y which gives us some nondegenerate hypersurface H_r within Y. Then we can cone this to a point in Z or a collect of points

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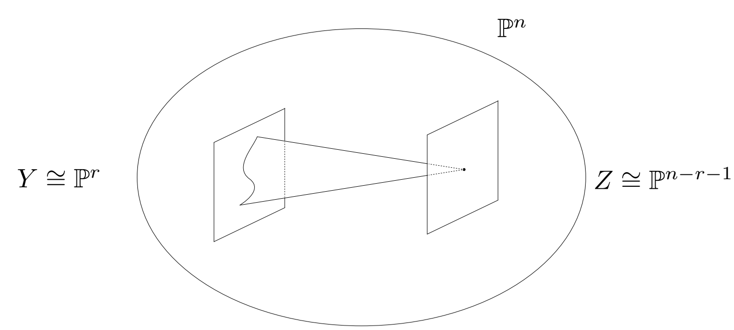

Figure 1. Two hypersurfaces Y and Z of $\mathbb{P}^n$. Then we take a hypersurface of Y and cone it with a point of Z.

in Z. When $r = n$, this is the nondegenerate case, and when $r < n$ this will be a cone over a nondegenerate quadric.

Example 1. Consider $r = n = 0$, then we just get the empty set.

Example 2. Consider $n > r = 0$. Clearly Y is empty, so if we cone over this, we don't get any lines, but we do get the points we are coning over. This means we get the cone over $\emptyset$ which is $\mathbb{P}^{n-1}$, or written as a hypersurface $H = V\left(x_0^2\right)$.

Example 3. If $r = n = 1$, so we're in $\mathbb{P}^1$, then we are putting the equation $x_0^2 + x_1^2$, or $H = (x_0 x_1)$ by changing coordinates, so $H = \{0, \infty\}$.

If $n > r = 1$, then $H =$ two $\mathbb{P}^{n-1}$s and f factors as $f = g_1 g_2$.

Example 4. If $r = n = 2$ we get a conic. Note that a conic in $\mathbb{P}^2$ always turns out to be isomorphic to $\mathbb{P}^1$.

Example 5. Consider $H \subset \mathbb{P}^n$ where

$$H = V\left(x_0 x_1 + x_2^2 + \cdots + x_n^2\right)$$

Now intersect H with U_0 by setting $x_0 = 1$, so we get the equation in affine coordinates $V\left(x_1 + x_2^2 + \cdots + x_n^2\right)$. So $x_1 = -x_2^2 - \cdots - x_n^2$, so this is the map $k^{n-1} \to k$ which sends

$$(x_2, \cdots, x_n) \mapsto -\left(x_2^2 + \cdots + x_n^2\right)$$

which is of course isomorphic to k^{n-1} by the projection.

2. Maps between classical varieties

First we have to talk a bit more about sheaves. Suppose we have a continuous map of topological space $\varphi : X \to Y$ and some presheaf A on X. Then we want to construct a presheaf on Y called the direct image (or push forward) $\varphi_* A$. We won't go into the details of it, but this construction will be functorial. Define:

$$(\varphi_* A)(U) \coloneqq A\left(\varphi^{-1}(U)\right)$$

with the obvious maps ρ_{UV}. If A is a sheaf, it follows that $\varphi_* A$ is a sheaf, since preimages preserves the condition we put on a sheaf.

Now a morphism of varieties $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous function $\varphi : X \to Y$ such that φ preserves regular functions in the following sense. For $\varphi : X \to Y$, there is a canonical map

$$\varphi_\flat : \operatorname{Fun}(Y, k) \to \varphi_* \operatorname{Fun}(X, k)$$

We want to send some $f \in \operatorname{Fun}(Y, k)(U)$, i.e. a map $f : U \to k$ to something in $(\varphi_* \operatorname{Fun}(X, k))(U) = \operatorname{Fun}(X, k)\left(\varphi^{-1}(U)\right)$, in particular

$$f \mapsto f \circ \left(\varphi|_{\varphi^{-1}(U)}\right)$$

So now the sense in which φ preserves these functions is that:

$$\varphi_\flat \mathcal{O}_Y \subseteq \varphi_* \mathcal{O}_X$$

Example 6. Let $X = k^{n+1} \setminus \{0\}$. Then $X = \bigcup_i W_i$ where $W_i = X_{x_i}$. There is a map $X \to \mathbb{P}^n$ which sends $(x_0, \cdots, x_n) \mapsto (x_0 : \cdots : x_n)$. This is certainly a map of sets, so we just need to check that this is a map of varieties. This map certainly maps each $W_i \to U_i \cong k^n$, so W_i is k^{n+1} minus a hyperplane. The coordinate ring on k^n is just $k[x_1, \cdots, x_n]$, and the coordinate ring on W_0 is just

$k\left[x_0^{\pm 1}, x_1, \cdots, x_n\right]$, and then the map on functions is just the obvious inclusions. This means this is a map of affine varieties, which is continuous. But this extends to the whole space, so this is indeed a continuous map, and a morphism. Because on an affine variety the global sections of the sheaf of functions is the same as the coordinate ring, it's fine to just check the maps from affine varieties to affine varieties to check that the morphism on the whole variety is a morphism. This also shows us $\mathbb{P}^n$ as the quotient of X by scalar multiplication.