LECTURE 11
MATH 256A

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1. Zariski topology

Fix a commutative ring R with unit. We want to equip this with a space $X = \operatorname{Spec} R$. As a set

$$X = \operatorname{Spec} R = \{P \subseteq R \mid P \text{ is prime.}\} \ .$$

Recall (1) is not prime. To give this a topology, we define the closed subsets to be

$$V(I) = \{P \mid I \subseteq P\}$$

The intention here is that for a ring homomorphism $R \to K$ to a field, we should have the following diagram:

$$\begin{array}{ccc} K & \longleftarrow & R \\ & & \\ \operatorname{Spec} K = \{\text{pt}\} & \longrightarrow & \operatorname{Spec} R \end{array}$$

where the bottom map should land in the kernel P of the ring homomorphism.

Proposition 1. *If we take $V(I)$ to be the closed sets, this is indeed a topology.*

Proof. First notice that $V((1)) = \emptyset$ and $X = V(0)$. Also

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$$

Now we claim that

$$V(I) \cup V(J) = V(I \cap J) = V(IJ)$$

Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, then the first inclusion $\subseteq$ is clear, and then since $IJ \subseteq I$ and $IJ \subseteq J$, then clearly $IJ \subseteq I \cap J$, so the second $\subseteq$ is clear. Now for any $P \supseteq IJ$, $f \in I$ and $g \in J$ means $fg \in P$, so either $g \in P$ or $f \in P$, so $I \subseteq P$ or $J \subseteq P$, so the furthest right is included in the furthest left, and this is indeed a topology. □

Example 1. Note that when R is a ring of functions, this is the classical Zariski topology we have already seen.

For $f \in R$, we write

$$X_f = X \setminus V(f) = \{p \in X \mid f \notin p\} \ .$$

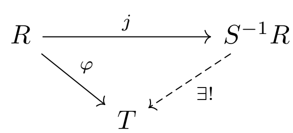
By construction, these form a base of the topology in the strong sense since $X_f \cap X_g = X_{fg}$.

Date: September 17, 2018.

2. Localization

Let $S \subseteq R$ be a multiplicative subset containing the unit of R. Now we want to construct a ring $S^{-1}R$ where we have inverted the elements of S. Note that there is no loss of generality in taking S to be multiplicatively closed, since if it wasn't and we inverted the elements, all of the products would be inverted as well. In general, for any R-module M we can form $S^{-1}M$ which is an $S^{-1}R$ modules.

In general, there's an almost tautological way to do this. We want a map j such that we have the following diagram:



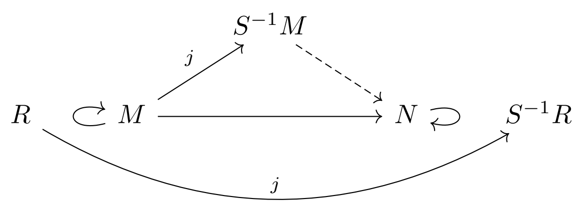
I.e. for any φ such that $\varphi(S) \subseteq T^{\times}$, j allows for the diagram to commute. We can explicitly write this as

$$S^{-1}R = R\left[u_s\right]_{s \in S} / \left(u_s s - 1 \mid s \in S\right)$$

Modules are just as easy since we can just take

$$S^{-1}M = S^{-1}R \otimes_R M$$

This is extension of scalars and is universal is the sense that:



where $j : a \mapsto 1 \otimes a$. This effectively follows for free from the universality of the tensor product.

This isn't the whole story because it is too abstract. But we do have the following explicit description:

Theorem 1. *We can write:*

$$S^{-1}M = \left\{ \frac{a}{s} \mid a \in M, s \in S \right\}$$

where a/s is really $s^{-1}j(a)$ and $a/s = b/t$ iff there exists some $v \in S$ such that $vta = vsb$ in M.

Proof. It is clear that if $vta = vsp$ then $a/s = b/t$, so we just have to show the opposite direction.

The first thing to verify is that there is indeed a well-defined R-module $\widehat{M}$ with elements $a^{(s)}$ such that $a^{(s)} = b^{(t)}$ iff there is some v such that $vta = vsb$.

There are two ways to do this. There is the boring way, where we take this to define an equivalence relation, and then check a bunch of things. Then there is the sort of magical way, where we form a direct system $\left(M^{(s)}\right)_{s \in S}$ where every element is a copy of M, and every arrow $M^{(s)} \to M^{(st)}$ is multiplication by t. Now we take the direct limit of this system. One thing we have to check is that this is in fact a direct system, which is true because of the following. For v such that $M^{(s)} \to M^{(v)}$

and $M^{(t)} \to M^{(v)}$, even though it might be the case that $v = ps = qt$ without being st, it is true that $v^2 = pqst$ so these all eventually map to $M^{(v^2)}$.

We claim that the map $a^{(t)} \to a^{(st)}$ is inverse to the multiplication $\cdot s$. So we need to check if $sa^{(st)} = a^{(t)}$ but $tsa = sta$ so we are done. Then by the universal property we get $S^{-1}M \to \widehat{M}$ where $s^{-1}a \mapsto a^{(s)}$ since $M \to \widehat{M}$ by $a \mapsto a^{(1)}$.

Finally we claim elements of the form $s^{-1}a \in S^{-1}M$ are the whole ring. By construction this is just $S^{-1}R \otimes_R M$, which means we just have to show that these elements contain the image of M. It is clear they are closed under addition just by addition of fractions, and it is also closed under multiplication by the elements u_s, so these elements form an $S^{-1}M$ submodule which contains M, so it is the whole thing.

We already saw the map is surjective, so we just need it to be injective. But if $s^{-1}a \mapsto 0$, then $a^{(s)} = 0$, which means there is some $v \in S$ such that $va = 0$ in M, which means clearly this is 0 in M as well, so it is injective. $\square$