LECTURE 12
MATH 265A

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1. More on localization

1.1. **Exactness.** Fix a ring R and a multiplicative subset S . Recall that we can invert S to get $S^{-1}R$, and similarly for a module M we can form $S^{-1}M = S^{-1}R \otimes_R M$, which has all of the universal properties we could want. But then it's difficult to describe, however we saw that

$$S^{-1}M = \left\{ \frac{a}{s} \mid a \in M, s \in S \right\}$$

and $a/s = 0$ iff $\exists v \in s$ such that $va = 0 \in M$. One thing we get immediately from this is that localization on modules is an exact functor. If tensoring with a module is exact, we say this module is flat.

First of all, if we have a homomorphism $\varphi : M \to N$, we get a homomorphism $S^{-1}M \to S^{-1}N$ where $a/s \mapsto \varphi(a)/s$. Going from $M \rightsquigarrow S^{-1}M$ is the functor $S^{-1}R \otimes -$. It is a general fact that the tensor product is always right exact, i.e. we have

$$\begin{array}{c} 0 \to K \to M \to Q \to 0 \\ \Big\downarrow{\scriptstyle A\otimes -} \\ A \otimes K \to A \otimes M \to A \otimes Q \to 0 \end{array}$$

Recall that if a functor preserves short-exact sequences it also preserves all exact sequences, so we just have to check what this functor does to injective maps. For $N \hookrightarrow M$ we get $S^{-1}N \to S^{-1}M$. But the image of something $a/s \in S^{-1}N$ maps to $0 \in S^{-1}M$ iff $va = 0 \in M$ for some $v \in S$, but this means that $va = 0 \in N$ as well since $a \in N$, so this does indeed preserve injective maps. Therefore $S^{-1}R$ is a flat R-module.

1.2. **Ideals.** Let $I \subseteq R$ be an ideal. We want to understand how this is related to $S^{-1}I \subseteq S^{-1}R$. This is the image of this under this exact functor described above.

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But we can also bring ideals in the other direction:

$$R \xrightarrow{j} S^{-1}R$$

$$a \longmapsto a/1$$

$$I \longrightarrow S^{-1}I$$

$$j^{-1}(Q) \longleftarrow Q$$

Definition 1. An ideal I is S-saturated if $sa \in I$ for $s \in S$ implies $a \in I$.

Proposition 1. *There is a bijection between ideals of $S^{-1}M$ and S-saturated ideals of M.*

Proof. We need to check that the preimage of Q (i.e. a such that $a/1 \in Q$) is S-saturated. If we take something in the preimage of Q, and act j on it, and then generate an ideal in $S^{-1}R$, certainly Q is contained in this ideal, but to see this is Q, we notice that anything in Q can be written $a/s \in Q$, so there is nothing extra added.

The other direction is basically the same story. Start with an ideal, see what its image under j generated, then the inverse image of this back in R will contain what you started with, but it might be larger. To see it is in fact equal, we notice that any $j(a) \in S^{-1}I$ can be written as $a/1 = b/s$ for some $s \in S$, which means there is some $v \in S$ such that $vsa = vb$, so $vsa \in I$, so $a \in I$. □

Any time we have a ring homomorphism $\alpha : R \to T$, if we have a prime ideal $Q \subseteq T$, then $P = \alpha^{-1}(Q)$ is prime since $R \to T \to T/Q$ is exact, so $R/P \hookrightarrow T/Q$. In this situation we have a map $\operatorname{Spec} T \to \operatorname{Spec} R$ which maps $Q \mapsto j^{-1}Q$. And if we think about the topology on these spaces, this is in fact a continuous map. In the classical case, if we look a point of $\operatorname{Spec} T$, this corresponds to a maximal ideal, and then the preimage is maximal as well, so this corresponds to the preimage of the initial point.

So we have seen that prime ideals in $S^{-1}R$ map to prime ideals of R, but it also works the other way around. For $P \subseteq R$ prime, if $P \cap S$ is empty, then P is S-saturated since $sx \in P$ implies s or x is in P, but the intersection with S is empty, so $x \in P$. Then we claim that this correspondence gives us a prime ideal $S^{-1}P \subseteq S^{-1}R$. We know j induces a homomorphism $R/P \to S^{-1}R/S^{-1}P$, but this is really just $S^{-1}R/S^{-1}P = S^{-1}(R/P)$. Since P is prime this is an integral domain, so inverting everything gives us the field of fractions, but if we invert a subset, such as $S^{-1}(R/P)$, then we get some subring of the field of fractions.

In conclusion we have the following canonical bijection:

$$\operatorname{Spec} S^{-1}R \cong \{P \in \operatorname{Spec} R \mid P \cap S = \emptyset\}$$

But there is a special case: for some $f \in R$, $R\left[f^{-1}\right] = S^{-1}R$ where $S = \left\{1, f, f^2, \cdots\right\}$. In this case,

$$\operatorname{Spec} R\left[f^{-1}\right] \cong \{P \in \operatorname{Spec} R \mid f \notin P\}$$

So for $X = \operatorname{Spec} R$, $X \setminus V(f) = X_f$. In addition, this correspondence turns out to yield a homomorphism $\operatorname{Spec} R\left[f^{-1}\right] \to X$.

Now we want to turn these topological spaces into spaces equipped with a sheaf of rings. In general, we even want to give it a sheaf of modules for that sheaf of rings on $\operatorname{Spec} R$. Even when $M = R$, the sheaf of rings is no longer a sheaf of functions. For sheaves of modules, even in the classic case we wouldn't expect this.

We have one example of a sheaf which is not a sheaf of functions, which is the direct image of a sheaf under a continuous map, but we will do something more general. It works for any sheaf. We can always think of it as a sub-sheaf of a sheaf of functions.

2. Stalks

For the moment this is just about sheaves on topological spaces. Let X be a topological space, and let $\mathcal{A}$ be a sheaf on X.[1]

Definition 2. For any $P \in X$, the *stalk* of $\mathcal{A}$ at P is the direct limit:

$$\mathcal{A}_P = \varinjlim_{U \ni P} \mathcal{A}(U)$$

Recall in a direct limit two things are equal if they map to the same things "later" in the direct limit. So two things represent the same element of this thing if they somehow agree eventually as open sets get smaller. For $p \in U$, and $a \in \mathcal{A}(U)$, $a \mapsto [a] \in \mathcal{A}_P$ is called the germ of a. Then $[a] = [b]$ for $a \in \mathcal{A}(U)$ and $b \in \mathcal{A}(V)$ iff there is some $P \subseteq W \subseteq U, V$ such that $a|_W = b|_W$.

For any sheaf, roughly speaking, we can map

$$\mathcal{A} \to \prod_p (i_p)_* \underline{\underline{\mathcal{A}_P}}$$

where $i_p : \{p\} \to X$ is just inclusion, and

$$\underline{\underline{\mathcal{A}_p}}(U) = \begin{cases} \{p\} & U = \{p\} \\ * & \text{otherwise} \end{cases}$$

where $*$ is the terminal object of the category which $\mathcal{A}$ takes values in. So a sheaf is always a subsheaf of this direct product of the so-called skyscraper sheaves. This is the sense in which any sheaf can be viewed as sort of being inside a larger sheaf of "functions."

[1] With values in any category with direct limits.