LECTURE 13
MATH 256A

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1. Examples

Consider a space X equipped with a sheaf $\mathcal{S}$. For any $P \in X$, recall that the stalk at P is defined to be:

$$\mathcal{S}_P = \varinjlim_{U \ni P} \mathcal{S}(U)$$

Concretely, for $x \in \mathcal{S}(U)$ such that $p \in U$, $x \mapsto [x] \in \mathcal{S}_p$, and $[x] = [y]$ for $y \in \mathcal{S}(v)$ iff there is some $p \in W \subseteq U \cap V$ such that $x|_W = y|_W$.

Example 1. Let A be a set.[1] Then we can form the constant sheaf $\underline{\underline{A}}$ on X which consists of locally constant functions $U \to A$. The stalks are what are really constant here since for any P, $\underline{\underline{A}}_P = A$.

Example 2. A locally constant sheaf is a sheaf which is locally isomorphic to a constant sheaf.

Example 3. If we want a sheaf on a one-point space $X = \{P\}$, there are only two open sets so we just need to specify $\mathcal{S}(X)$ and $\mathcal{S}(\emptyset)$. By the sheaf axiom, $\mathcal{S}(\emptyset) = \{0\}$ must be the terminal object, i.e. the identity with respect to forming products. But $\mathcal{S}(X) = A$ can be anything, so the collection of such sheaves is equivalent to the category $\mathcal{S}$ takes values in. The single stalk is just $\mathcal{S}_P = A$.

Example 4. For $P \in X$ we can embed $i_P : \{P\} \to X$ and then $(i_P)_* \underline{\underline{A}} = \mathcal{S}$ is a sheaf on X called the skyscraper sheaf. In particular it assigns

$$\mathcal{S}(U) = \begin{cases} A & P \in U \\ \{0\} & P \notin U \end{cases}$$

and the stalks are

$$\mathcal{S}_Q = \begin{cases} A & Q \in \overline{\{P\}} \\ \{0\} & Q \notin \overline{\{P\}} \end{cases}$$

so $S_P = A$.

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[1] Or module, ring, etc.

2. Constructing a sheaf from its stalks

Let's see how sheaves are related to their stalks. Let $\mathcal{S}$ be a sheaf on X with stalks $\mathcal{S}_P$. Then form a sheaf:

$$\tilde{\mathcal{S}} = \prod_P (i_P)_* \underline{\underline{\mathcal{S}_P}}$$

The sections of this sheaf are:

$$\tilde{\mathcal{S}}(U) = \prod_{p \in U} \mathcal{S}_P$$

this is called a flasque sheaf. They can roughly be thought of as sort of flexible. Note that we can map $\mathcal{S} \to \tilde{\mathcal{S}}$ by mapping

$$x \mapsto \{[x]_P \in \mathcal{S}_P\}$$

The sheaf axiom implies that any section $x \in \mathcal{S}(U)$ is determined by its germs. This means that sending a section to a list of its germs as above is actually injective.

For any sheaf, this means we can represent it as a subsheaf of a flasque sheaf. So the idea is to sort of do this without knowing $\mathcal{S}$ in the first place. So if we only have the stalks, we can form $\tilde{\mathcal{S}}$, and as long as we use a local condition to find a subsheaf, this will automatically be a sheaf.

2.1. **The case of** $\operatorname{Spec} R$**.** Consider a ring R and R-module M. Let $X = \operatorname{Spec} R$. We want to build a sheaf $\tilde{M}$ on X. Recall that for $f \in R$, $X_f = X \setminus V(f)$ is canonically homeomorphic to $\operatorname{Spec} R\left[f^{-1}\right] = \operatorname{Spec} R_f$.

For P a prime ideal, by definition $S \coloneqq R \setminus P$ is a multiplicative set. So $R_P \coloneqq S^{-1}R$. This is then a local ring, so it has a unique maximal ideal which is in fact PR_p. We can also define $S^{-1}M = M_P$. The key property of the sheaf $\tilde{M}$ will be that the stalk at each point will be M_P.

The sections of $\tilde{M}(U)$ will be elements of

$$\prod_{p \in U} M_p$$

which, locally on X_f, are of the form a/f for $a \in M$ and $f \in R$. If M_f is an R_f module, then for $P \in X_f$, P is a prime of R_f.

Now we want to see if this is really a sheaf. Consider $a/f, b/g \in M_P$ (where $f, g \notin P$) such that $a/f = b/g$ in M_P. Then the question is whether or not this implies that $a/f = b/g$ in M_Q for Q in a neighborhood of P. Recall $a/f = b/g$ in M_P just means $\exists h \notin P$ such that $hga = hfb$ in M. But if this is the case, then $a/f = b/g$ on X_{fgh} so we are done.

If $M = R$, then $\tilde{R}$ is a sheaf on X with stalks R_P and a priori it's just a sheaf of R-modules. However as it turns out, we can multiply pointwise, so there's really a sheaf of rings $\mathcal{O}_X$, and then it is a sheaf of modules over this sheaf of rings.

So from R we get a space and a sheaf of rings on it $(X, \mathcal{O}_X)$, and we know the stalks, and in some sense the sections $\mathcal{O}_{x,P} = R_P$. Similarly, for any module M we can get $\tilde{M}$ which is a sheaf of $\mathcal{O}_X/\mathbb{C}$-modules.

One of our first results will be that in fact

$$\mathcal{O}_X(X_f) = R_f \qquad \tilde{M}(X_f) = M_f$$