LECTURE 14

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Luya's notes

1. Radical of an ideal

Recall the notion of a radical of an ideal.

Lemma 1. *For R a commutative ring and $I \subseteq R$ an ideal,*

$$\sqrt{I} = \bigcap_{P \supseteq I} P$$

Proof. Recall $x \in \sqrt{I}$ iff for some m $x^m \in P$. Then if $I \subseteq P$, $x^m \in P$ implies $x \in P$ so we have this containment.

Now if $x \notin \sqrt{I}$, consider the ring $(R/I)\left[x^{-1}\right]$. Pick a maximal (so prime) ideal $Q \subseteq R/I\left[x^{-1}\right]$. Since we have a ring homomorphism:

$$\alpha : R \to (R/I)\left[x^{-1}\right]$$

we can take the preimage $P = \alpha^{-1}Q$, and then $I \subseteq P$, and $x \notin P$ implies

$$x \notin \bigcap_{P \supseteq I} P$$

as desired. □

Let $X = \operatorname{Spec} R$. Recall that $I \subseteq R$ gives us:

$$V(I) = \{P \supseteq I\} \subseteq \operatorname{Spec} R$$

and $Y \subseteq X$ gives us:

$$\mathcal{I}(Y) = \{f \in R \,|\, Y \subseteq V(f)\} = \bigcap_{P \in Y} P$$

which is an ideal, and in particular

$$\mathcal{I}(Y) = \sqrt{\mathcal{I}(Y)}$$

Consider $V(\mathcal{I}(Y))$.

Claim 1. If $Q \in Y$, then

$$Q \supseteq \bigcap_{P \in Y} P = \mathcal{I}(Y)$$

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Tautologically, $Y \subseteq V(\mathcal{I}(Y))$.

Now let $Z = V(J) \supseteq Y$. Then for any $f \in J$, $Y \subseteq V(f)$, so $f \in P$ for all $P \in Y$ which implies

$$f \in \bigcap_{P \in Y} P = \mathcal{I}(Y)$$

Therefore, $J \subseteq \mathcal{I}(Y)$, so

$$Z = V(J) \supseteq V(\mathcal{I}(Y))$$

This means $V(\mathcal{I}(Y))$ somehow sandwiches itself in between Y and anything that contains it, so this is the closure of Y, $\overline{Y}$.

In the other direction:

$$\mathcal{I}(V(I)) = \bigcap_{P \supseteq I} P = \sqrt{I}$$

this is the "trivial nullstellensatz" we were promised. Note that this gives us the association:

$$\left\{I \subseteq R \,|\, I = \sqrt{I}\right\} \cong \left\{Z \subseteq \operatorname{Spec} R \,|\, Z = \overline{Z}\right\}$$

where we associate I to $V(I)$ and Z to $\mathcal{I}(Z)$. I.e. irreducible Z gets associated to prime I and vice versa.

Proof. First let I be prime. Then if $V(I) = V(J_1) \cup V(J_2)$ this means $V(I) = V(J_1 J_2)$. But then

$$\mathcal{I}V(I) = \sqrt{I} = I = \mathcal{I}V(J_1 J_2) \supseteq J_1 J_2$$

but if $J_1 J_2 \subseteq I$, this means either $J_1 \subseteq I$ or $J_2 \subseteq I$. WLOG let $J_1 \subseteq I$, then $V(J_1) \supseteq V(I)$.

If I isn't prime, then for $fg \in I$ either $f \notin I$ or $g \notin I$. If we define:

$$Z = V(I) \qquad Z_1 = Z \cap V(f) \qquad Z_2 = Z \cap V(g)$$

we have that $Z \subseteq V(f) \cup V(g)$ which means $Z = Z_1 \cup Z_2$ is not irreducible. $\square$

Note that points don't have to be closed, and ideals don't have to be maximal.

$$V(P) = V(\mathcal{I}(\{P\})) = \overline{\{p\}} =: \overline{P}$$

2. Examples

Example 1. For R a local ring, it has a unique closed point associated to the unique maximal ideal.

Example 2. Let $R = k[x_1, \cdots, x_n]$ for $k = \overline{k}$. Note that classically $R = \mathcal{O}(k^n)$. Now the above result gives us an association of $X = \operatorname{Spec} R$, with irreducible closed subsets $Z \subseteq k^n$. Note that we have a homeomorphism between $k^n \leftrightarrow X_{cl}$, and a quasi-homeomorphism between $k^n \hookrightarrow X$.

Recall that we have:

$$X_f = X \setminus V(f) \qquad (X_{cl})_f = (X_f)_{cl} \ .$$

We have a sheaf $\mathcal{O}_Y$ of k-valued functions on Y locally of the form a/f, and then on the other side we have $\mathcal{O}_X = \tilde{R}$. Because the sections of this are effectively things locally of the form a/f as well, these sheaves are actually the same.

$$\mathcal{O}_X = \tilde{R} = \mathcal{O}_{X_{cl}}$$

Example 3. For $I \subseteq R$ consider $\operatorname{Spec} R/I$ and $\operatorname{Spec} R$. We have a ring homomorphism

$$R \xrightarrow{\alpha} R/I$$

$$\alpha^{-1}(P) \dashleftarrow P$$

where $\alpha^{-1}P$ is prime as well, which means we have a map $\varphi : \operatorname{Spec} R/I \to \operatorname{Spec} R$ which is continuous because

$$\varphi^{-1}(V(J)) = V(\alpha(J))$$

so $i : V(I) \hookrightarrow X = \operatorname{Spec} R$.

For the module $M = R/I \leftarrow R$, we write $M_{(R)}$ for M as an R-module. Then

$$I\left(M_{(R)}\right)_P = \begin{cases} 0 & I \not\subseteq P \\ M_P & I \subseteq P \end{cases}$$

$$\tilde{M}_{(R)}(U) = \tilde{M}(U \cap V(I))$$

i.e.

$$\tilde{M}_{(R)} = i_* \tilde{M} \qquad\qquad \widetilde{R/I} = i_* \mathcal{O}_Z$$

where $Z = \operatorname{Spec} R/I$ so we get $(Z, \mathcal{O}_Z)$.