LECTURE 15

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1. Review/extension of last time

Example 1. Let's consider a classical example. Let

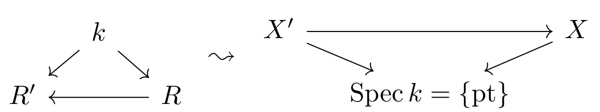
$$R = \mathcal{O}(Y) = k[X_1, \cdots, X_n] / \mathcal{I}(Y) \ .$$

Then we want to know what $X = \operatorname{Spec} R$ looks like. Among the points of X are of course the maximal ideals which correspond to classical points $X_{cl} \cong Y$, but also some extra points P such that $\overline{\{P\}} = Z$ is any irreducible closed subvariety of Y. So any irreducible closed subset of Y is the closure of some point.

The inclusion $X_{cl} \hookrightarrow X$ is a quasi-homeomorphism so it induces a bijection of closed subsets obviously, but this of course means it also gives a bijection of open subsets. This means Y and X somehow have the same topology on "different" points. This means $\mathbf{Sh}(X) \cong \mathbf{Sh}(Y)$ are isomorphic.[1]

We have a sheaf $\mathcal{O}_Y$ of k-valued functions on Y locally of the form a/f, and then on the other side we have $\mathcal{O}_X = \tilde{R}$. Because the sections of this are effectively things locally of the form a/f as well, these sheaves are actually the same.

We will see much more of this. For example, recall we have that $Y' \to Y$ corresponds to $R = \mathcal{O}(Y) \to R' = \mathcal{O}(Y')$ is a k-algebra homomorphism. Similarly, we will see that $R \to R'$ will give us a homomorphism $X' = \operatorname{Spec} R' \to X = \operatorname{Spec} R$. And if $R \to R'$ is a k-algebra homomorphism, we get that:



Example 2. Take any ring, and quotient it by an ideal $\pi : R \to R/I = S$. Then let $Z = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$. Of course this means we have a map $i : Z \to X$ where $P \mapsto \pi^{-1} P$. Now we can analyze things very hands on. The ideals, prime or not, of S correspond to ideals of R containing I, so of course this is true for prime ideals as well. So the map $Z \to X$ is a homeomorphism $Z \simeq V(I) \subseteq X$.

We can also say how the sheaves of rings attached to these are related. Consider some S-module M. Of course we can also think of this as an R-module $M_{(R)}$ using the map π. On Z we have a sheaf $\tilde{M}$, and on X we have a sheaf $\tilde{M}_{(R)}$. For any $P \in \operatorname{Spec} R$, since any element of I kills M we have

$$\left(M_{(R)}\right)_P = \begin{cases} 0 & P \notin V(I) \\ \left(M_Q\right)_{(R_p)} & P \in V(I) \end{cases}$$

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[1] Note these aren't just equivalent categories, they are actually isomorphic.

where $Q = P/I$. So the relationship between these two sheaves is:

$$i_* \mathcal{O}_Z = \mathcal{O}_X / \tilde{I}$$

Example 3. Again, let $X = \operatorname{Spec} R$ for any ring R. Now we want to see what $\operatorname{Spec} R_f$ looks like. We know there is a canonical homomorphism $R \to R_f$ which gives us a map $\operatorname{Spec} R_f \to X$. We know ideals of R_f are just f-saturated ideals of R, so prime ideals of R_f are prime ideals of R which don't contain f. Therefore this map $\operatorname{Spec} R_f \to X_f = X \setminus V(f) \subseteq X$ is a bijection. Of course this is an open subset, and this map a homeomorphism onto this open subset.

The sheaves are related as follows. We have the sheaf $\tilde{M}$ associated to any R-module on X, but we also have the sheaf $\tilde{M}_f$ which gives us a sheaf on $\operatorname{Spec} R_f \cong X_f$. So for any $P \in X_f$, $(M_f)_P = M_P$, so the stalks are the same at any point of X_f, and the germs are the same, so all of the sections are the same. Therefore the relationship is that $\tilde{M}_f = \tilde{M}|_{X_f}$. Taking $M = R$, this gives us $\mathcal{O}_X|_{X_f} = \mathcal{O}_{X_f}$.

Example 4. Let's look at the beginning of the relationship between this story and number theory. Consider $\operatorname{Spec} \mathbb{Z}$. $\mathbb{Z}$ is an integral domain, so (0) is prime, and the only other primes are (p) for p prime. The topology is as follows. For $I = (n)$, the only primes that contain it are (p) for $p|n$. So this is somehow a line, where every prime gives us a maximal ideal, so they correspond to closed points, and then we have an extra point, (0), which has closure the entire space.

In this picture we should think of any integer as a function, since it defines a global section of the sheaf of functions on here. The stalks are just localizations since

$$\mathbb{Z}_{(p)} = \{ r/s \,|\, p \nmid s \}$$

and $\mathbb{Z}_{(0)} = \mathbb{Q}$. The only thing that's weird, is that this function has values in a finite field $\mathbb{Z}_{(p)} / (p) \mathbb{Z}_{(p)} = \mathbb{Z}/p\mathbb{Z}$.

2. The next goal

Now that we have some examples we will fill in the theory. so to any ring, we have a space with sheaf of rings, and to any module a sheaf of modules. But we only specified these using stalks. This is why the above examples only consider $R \to R/I$ and $R \to R_f$ instead of arbitrary ring homomorphisms. However it is indeed try that for any ring homomorphism $R \to S$, an S-module will give a morphism of affine schemes.

To see this, we want to describe $\tilde{M}$ in terms of its sections, in particular we want to describe its global sections. Recall $\operatorname{Spec} R_f = X_f$, and $\tilde{M}_f = \tilde{M}|_{X_f}$. In particular it turns out that $\tilde{M}(X) = M$, and it will be a corollary that $\tilde{M}(X_f) = M_f$. Then since all such open sets form a base, this is actual a full description. We will prove and use the following lemma next time:

Lemma 1. *If $a \in M$ has $a/1 = 0$ in M_P for all $P \in \operatorname{Spec} R$, then $a = 0$.*

In other words, sections of a module are determined by their germs in the localizations of that module.