LECTURE 16
MATH 256A

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1. Preliminary results regarding localization of modules

Let $X = \operatorname{Spec} R$ and M be an R-module. Recall the construction $\tilde{M}$. We want to show that the map $M \to \tilde{M}(X)$ is an isomorphism. First we have a lemma that effectively says this is injective:

Lemma 1. *If $x \in M$ has $0 = x_p \in M_p$, where $x_p = x/1$ is the image of x in M_p, then $x = 0$ in M.*

Proof. Let

$$I = \operatorname{Ann}(x) := \{f \in R \mid fx = 0\}$$

For $x \neq 0$, $I \neq (1)$, so it is contained in some maximal ideal $P \supseteq I$, which is of course prime. Now consider $x_P = 0 \in M_P$. Since in X_P we have inverted $S = R \backslash P$, we have that for all $s \in S$, $sx \neq 0$, so in fact $x_P \neq 0$. □

Corollary 1. *If $x_P = 0$ for all $P \in (\operatorname{Spec} R)_{cl}$, then $x = 0$.*

2. Support

For X any space and $\mathcal{M}$ a sheaf of abelian groups, we define the support of a section $x \in \mathcal{M}(U)$ to be

$$\operatorname{Supp}(x) = \{P \in U \mid x_P \neq 0\}$$

But in the situation above, $x_P = 0$ just means there is some neighborhood $V \ni P$ such that $x|_V = 0$. Therefore the support of x is always closed in U.

We can also talk about the support of a sheaf itself:

$$\operatorname{Supp} \mathcal{M} = \{P \mid \mathcal{M}_P \neq 0\}$$

This is not closed in general.

Example 1. Let $\mathcal{M}$ be $\tilde{M}$ on $\operatorname{Spec} R$ where $M = \mathbb{R}\{x_1, \cdots, x_k\}$ is finitely generated. The support of any section x is just $V(\operatorname{Ann}(x))$. Similarly, the support of M is

$$\operatorname{Supp}\left(\tilde{M}\right) = \bigcup_i \operatorname{Supp}(x_i) = \bigcup_i V(\operatorname{Ann}(x_i)) = V(\operatorname{Ann}(M))$$

so it is in fact closed.

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3. Quasi-compactness

Usually compactness is reserved for Hausdorff spaces, so we will define a space Y to be quasi-compact iff every open cover has a finite subcover. In particular, if Y is also Hausdorff, then it is compact.

Fact 1. *$X = \operatorname{Spec} R$ is quasi-compact.*

Proof. Consider an open cover. Since any element of this cover can be covered by elements of the base of the Zariski topology, WLOG we can just consider the cover as:

$$X = \bigcup_{\alpha} X_{f_\alpha}$$

which means

$$\emptyset = \bigcap_{\alpha} X_{f_\alpha} = V((f_a))$$

But from the trivial nullstellensatz, this happens iff $(f_\alpha, \cdots) = (1)$. But this means

$$1 = g_1 f_{\alpha_1} + \cdots + g_n f_{\alpha_n}$$

but then this finite collection $\{f_{\alpha_i}\}_{i=1}^{n}$ generates (1), so the sets $V(f_{\alpha_i})$ for $i = 1$ to n cover X. $\square$

4. Noetherian spaces

We will call a space Noetherian if there is no infinite strictly decreasing chain of closed subsets. This is the sort of contravariant version of a ring being Noetherian. In particular:

Proposition 1. *If R is a Noetherian ring, $\operatorname{Spec} R$ is a Noetherian space.*

The converse is false since $\operatorname{Spec} R$ doesn't know about all of the ideals of R, it only knows the radical ideals, so we could have an infinitely increasing chain of ideals which all have the same radical.

Lemma 2. *Y is a Noetherian space iff there are no strictly increasing chains of open subsets iff every open subset is quasi-compact.*

This lemmas gives us a reason to not consider only Noetherian rings. The point is, $\operatorname{Spec} R$ is always quasi-compact, but every open subset doesn't have to be unless R is Noetherian in the first place. This tells us to look at non-Noetherian rings for examples of this:

Example 2. Consider a non-Noetherian ring such as:

$$R = k[x_1, x_2, \cdots]$$

Then $X = \operatorname{Spec} R$ can be thought of as infinite dimensional affine space. Now $V(x_1)$ is like a hyperplane, and then we can keep insisting on additional coordinates being zero, so we found a strictly decreasing infinite chain of closed subsets:

$$X \supset V(x_1) \supset V(x_1, x_2) \supset \cdots$$

Then the intersection of them all is $V(x_1, \cdots) = \operatorname{Spec} k = \{0\}$. So the complement of these things is a strictly increasing chain of open subsets, and the union of these consists of everything except the origin. But $X \setminus \{0\}$ can be covered by this sort of infinite cover, which has no finite subcover. So we found a non-quasi-compact open set.

We can define an infinite dimensional projective space:

$$\mathbb{P}^\infty = \left(\mathbb{A}^\infty \setminus 0\right)/k^\times$$

which is perfectly good as a projective space, but is not quasi-compact.

5. Main theorem

Theorem 1. *The map $M \to \tilde{M}(X)$ is an isomorphism.*

Proof. Lemma 1 tells us this is injective, so we effectively just have to show surjectivity. Given $\alpha \in \tilde{M}(X)$, (recall this means $\alpha_P \in M_P$) there is some covering, (which we can take to be finite by quasi-compactness):

$$X = \bigcup_i X_{f_i}$$

such that $\alpha = a_i/f_i$. This should really be a power of f_i on the bottom, but $X_{f_i^m} = X_{f_i}$ so WLOG we can just write this as a single power. Writing $\alpha = a_i/f_i$ is somewhat subtle, and a priori might not be true for all i, but lemma 1 tells us it's fine. Now we have:

$$(a_1/f_1, \cdots, a_n/f_n) \in \bigoplus_i M_{f_i}$$

which is in the kernel of the map:

$$\bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j}$$

where we take the difference of elements of M_{f_i} and elements of M_{f_j} to get an elements of $M_{f_i f_j}$. So saying something is 0 is saying the difference in the components is 0. In fact, everything in the kernel represents a section of M. But of course we are trying to show that any given section comes from an element of M.

But there is another map:

$$M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j}$$

so it is enough to show that this sequence is exact at the middle term. As it turns out, the first map turns out to be injective from lemma 1, and in fact this sequence continues to give what is called the Čech complex, which we claim is exact.

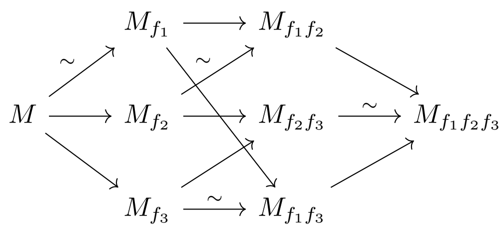
Claim 1. The following complex is exact:

$$M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \bigoplus_{i<j<k} M_{f_i f_j f_k} \to \cdots \to \bigoplus M_{f_1 \dots f_n} \to 0$$

We won't give the details of the additional maps, since there are some fiddly details regarding signs. The point is, for example, that three things of the form $M_{f_i f_j}$ map to the same $M_{f_i f_j f_k}$, and you take their sum, only you need to choose signs.

To see this sequence is exact, it is enough to show it after localizing each of the objects at prime P. This is because lemma 1 tells us that localization is exact. Therefore we will consider the homology groups of the resulting localized complex, and these will be the localizations of the homology of the original complex, so it is enough to just check that the homology of the localized sequence is 0.

But now this is easy to see, because it's made of little exact sequences. In particular, every time we add an f_1 to the index set, the corresponding map is an isomorphism, which is just an exact sequence with two adjacent nonzero objects as in the following diagram:



Since direct sums of exact complexes are still exact, even if you tweak the maps, the desired complex is exact. □

Remark 1. In the proof of the preceding theorem we only needed to see exactness at the first object. However as we will see later when we study sheaf cohomology, the exactness means something at each of the objects in the complex.

Next time we will see that $\tilde{M}(X_f) = M_f$ as a corollary of this theorem.