LECTURE 17
MATH 256A

LECTURE BY: PROFESSOR MARK HAIMAN
NOTES BY: JACKSON VAN DYKE

1. Affine schemes

So at this point, we have an effectively complete definition of $\operatorname{Spec} R$. This is a topological space with a sheaf of rings $\mathcal{O}_x = \tilde{R}$, and if we have an R-module M, we have a sheaf of $\tilde{M}$ modules over $\mathcal{O}_X$. We have a pretty good handle on this, since by definition we have the stalks $\tilde{M}_P = M_P$. That is, if we look at things locally, i.e. in $X_f \supseteq U \ni P$, they should be of the form a/f for $a \in M$ and $f \in R$. Last time we saw that $\tilde{M}(X) = M$, and as a corollary $\tilde{M}(X_f) = M_f$.

When is $X_g \subseteq X_f$? This is just saying $V(f) \subseteq V(g)$. We already saw on $\operatorname{Spec} R$, one closed subset will be contained in another iff the radicals of the corresponding ideals are contained in one another. So this is the same as $g \in \sqrt{(f)}$, i.e. for some m and some $a \in R$, $g^m = af$. So if g has an inverse, so does g^m, which means af does, which means f does.

So what is the restriction map: $\tilde{M}(X_f) \to \tilde{M}(X_g)$? This is just a map $M_f \to M_g$. We know we have a localization map $M \to M_g$, but the universal property of M_f is that if we have a map from M to any R_f module, it will factor through. But f is invertible from the above discussion, so M_g is such an R_f module, and therefore we have the following diagram:

$$\begin{array}{ccccc} & & M_f & & \\ & \nearrow & & \searrow & \\ M & & \longrightarrow & & M_g \end{array}$$

This is what we will call an affine scheme. A classical affine variety is an affine scheme, as long as we take R to be our ring of functions.

2. Local rings

A commutative ring with unit A will be a local ring iff it has a unique maximal ideal.

Example 1. The zero ring is not local, since a maximal ideal must be proper.

Example 2. Fields are local rings since the zero ideal is the unique maximal ideal.

Lemma 1. *A ring is local iff every $a \in A \setminus \mathfrak{m}$ has an inverse.*

Proof. Let A be a local ring. Then take an element a outside of the maximal ideal $\mathfrak{m}$. This means any ideal containing a must be the unit ideal, which means a is invertible. Conversely, assume every non-unit of A is contained in a maximal ideal.

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Then this ideal must be maximal, since adding any additional element to this ideal makes it into the unit ideal since any additional element is invertible. □

Example 3. For $P \subseteq R$ a prime ideal, R_P is a local ideal. The unique maximal ideal is PR_P. Indeed anything not in this was inverted by inverting $S \coloneqq R \setminus P$. The residue field is the field of fractions of R/P, which is just $R_P/(PR_P)$.

3. Ringed spaces

3.1. **Definitions and examples.** A space $(X, \mathcal{O}_X)$ is called a *ringed space* iff $\mathcal{O}_X$ is a sheaf of rings. It is a *locally ringed space* iff the stalks $\mathcal{O}_{x,P}$ are local rings for all $P \in X$.

Then an affine scheme is a ringed space which is isomorphic as a ring space to $\operatorname{Spec} R$.

Now define an arbitrary scheme to be a ringed space which can be covered by open sets which are affine scheme.[1] Similarly observe that all schemes are locally ringed spaces.

To get anywhere we will have to talk about morphisms of ringed spaces, but we will introduce some simple examples first.

Example 4. An affine scheme is an example of a ringed space since every stalk of an affine scheme is a ring localized at a prime, and therefore a local ring.

Example 5. Any open subset $U \subseteq X$ of a scheme is a scheme. To see this, notice that since affine schemes comprise a basis for X, any open subset of X can be covered by affine schemes.

Example 6. Classical varieties Y over $k = \overline{k}$ are schemes. We still have to prove some things to see this, but it is true. Y will be the set of closed points X_a for a scheme naturally related to it, and when you do this, there are a few things to notice. First, Y is basically the same as X since the inclusion of the set of closed points into X is a quasi-homeomorphism. That is, the topology hasn't fundamentally changed.

In fact the category of classical varieties over k is the same as the category of reduced[2] schemes locally of finite type over $\operatorname{Spec} k$.

Example 7. Classical affine space was just k^n, but we can think of this as Spec of the polynomial ring $k[x_1, \cdots, x_n]$. Of course we can do this over any ring R, which will come with a free map $\operatorname{Spec} R[x_1, \cdots, x_n] \to \operatorname{Spec} R$ which is just given by inclusion. This is called affine n-space over R, written $\mathbb{A}^n(R) = \operatorname{Spec} R[x_1, \cdots, x_n]$. We can then form the following fiber product:

$$\begin{array}{ccc} k^n & \dashrightarrow & \mathbb{A}^n(R) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} R \end{array}$$

Similarly, we can define projective space $\mathbb{P}^n(R)$ over R. This will have the same property, where the geometric fibers over the map $\mathbb{P}^n(R) \to \operatorname{Spec} R$ will look like the usual n-dimensional projective space over k.

[1] This definition is analogous to our definitions of affine varieties and generic varieties.

[2] This means there are no nonzero nilpotent elements in the structure sheaf.

We will eventually see that it's somehow true that the only case we will need is $R = \mathbb{Z}$. $\mathbb{Z}$ universally maps to any ring, which means every scheme X is somehow canonically living over $\operatorname{Spec}\mathbb{Z}$: $X \to \operatorname{Spec}\mathbb{Z}$.

3.2. **Morphisms of ringed spaces.** It is easy to define morphisms of ringed spaces if the sheaf of rings is a sheaf of functions, but we have to work harder to define this for generic ringed spaces. To do this, we need some more sheaf theory.

Consider a continuous map $f : X \to Y$ between topological spaces, and a sheaf $\mathcal{S}$ on X. Then we know we can define $f_*\mathcal{S}$ by taking

$$f_*\mathcal{S}(U) = \mathcal{S}\left(f^{-1}(U)\right)$$

This actually makes sense for presheaves.

Remark 1. There are two basic types of operations on (pre)sheaves. The first type is somehow defined in terms of sections, is left exact, and makes sense for presheaves as well. The second type is somehow defined in terms of stalks, is right exact, and doesn't make sense for presheaves. The above is an example of the first type, and the next operation is an example of the second type.

Now we want to take a sheaf $\mathcal{A}$ on Y, and define a sheaf $f^{-1}\mathcal{A}$ on X. Let $P \in X$ be a point of X and define

$$\left(f^{-1}\mathcal{A}\right)_P \coloneqq \mathcal{A}_{f(P)}$$

The sections of $f^{-1}\mathcal{A}$ on U will be contained in

$$\prod_{P \in U} \left(f^{-1}\mathcal{A}\right)_P$$

They will satisfy the following property. Let $V \supset f(P)$ be an open set, then $U = f^{-1}(V)$ is an open set containing P. Now pick a section $s \in \mathcal{A}(V)$. Then we will insist that the sections of $f^{-1}\mathcal{A}(U)$ are locally of the form $f^{-1}s$ on $W \subseteq U$. Now this gives us a sheaf since these are local conditions, which are also clearly compatible with restrictions.

Example 8. If $f : X \to X$ is the identity and $\mathcal{A}$ is any presheaf on X, $f^{-1}\mathcal{A}$ is a sheaf on X. This is then called the sheafification.

As it turns out,

$$f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y) \qquad\qquad f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$$

are adjoint functors. The direct image functor is right adjoint to the inverse image functor. This means that for a continuous map $f : X \to Y$ and for sheaves $\mathcal{S}$ on X and $\mathcal{A}$ on Y,

$$\operatorname{Hom}_{\mathbf{Sh}(X)}\left(f^{-1}\mathcal{A}, \mathcal{S}\right) = \operatorname{Hom}_{\mathbf{Sh}(Y)}\left(\mathcal{A}, f_*\mathcal{S}\right)$$