LECTURE 18
MATH 256A

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1. Adjunction

Consider a continuous map $f : X \to Y$ and let $\mathcal{A}$ be a sheaf on X, and $\mathcal{B}$ be a sheaf on Y. Now we can form a sheaf $f_* \mathcal{A}$ on Y, which is defined to have sections:

$$f_* \mathcal{A}(V) = \mathcal{A}\left(f^{-1}(V)\right)$$

and similarly we can form $f^{-1}\mathcal{B}$ on X, which is defined to have the following stalks:

$$\left(f^{-1}\mathcal{B}\right)_p = \mathcal{B}_{f(p)}$$

But this doesn't tell us anything about the sections. So we say that the sections are locally of the form $f^{-1}(s)$ where s is the section of some open set of Y, and explicitly if we have $s \in \mathcal{B}(V)$, then

$$f^{-1}s \in \left(f^{-1}\mathcal{B}\right)\left(f^{-1}(V)\right) \subseteq \prod_{p \in f^{-1}(V)} \mathcal{B}_{f(p)}$$

and in particular

$$f^{-1}s = p \mapsto s_{f(p)}$$

The moral is that the direct image functor preserves sections and the inverse image functor preserves stalks.

Remark 1. We can do either construction for a presheaf, and the direct image will still be a presheaf, but the inverse image gives you an actual sheaf.

Theorem 1. *f^{-1} is left adjoint to f_*. Equivalently, f_* is right adjoint to f^{-1}.*

Proof. This just means if we have a map

$$\varphi : \mathcal{B} \to f_* \mathcal{A}$$

which is an arrow in $\mathbf{Sh}(Y)$, then we have a functorial bijection between such maps and maps

$$\psi : f^{-1}\mathcal{B} \to \mathcal{A}$$

which is of course an arrow in $\mathbf{Sh}(X)$.

First suppose we are given φ. Let $p \in X$ map to $f(p) = q \in Y$. Let $q \in V \subseteq Y$ be an open neighborhood of q, so $U = f^{-1}V$ is an open neighborhood of p. Then for any section $s \in \mathcal{B}(V)$, we can map this to $\varphi(s)$, which is of course in $(f_*\mathcal{A})(V)$. But by definition, this is just $\mathcal{A}(U)$, so we can map this to $\varphi(s)_p \in \mathcal{A}_p$. Alternatively, starting with our original section $s \in \mathcal{A}(V)$ we can map this to $s_q \in \mathcal{B}_q = \left(f^{-1}\mathcal{B}\right)_p$.

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Then we can map this to $\varphi(s)_q \in (f_* A)_q$. But now there is a unique map from this to $\varphi(s)_p$. I.e. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(V) & \longrightarrow & \mathcal{B}_q = \left(f^{-1}\mathcal{B}\right)_p \\ \downarrow & & \downarrow \\ (f_*\mathcal{A})(V) = \mathcal{A}(U) & & (f_*\mathcal{A})_q \\ \downarrow & \swarrow & \\ \mathcal{A}_p & & \end{array} \qquad \begin{array}{ccc} s & \longmapsto & s_q \\ \downarrow & & \downarrow \\ \varphi(s) & & \varphi(s)_q \\ \downarrow & \swarrow & \\ \varphi(s)_p & & \end{array}$$

Now if we are given $\psi : f^{-1}\mathcal{B} \to \mathcal{A}$, then for $s \in \mathcal{B}(V)$,

$$\begin{array}{c} f^{-1}s \in \left(f^{-1}\mathcal{B}\right)(U) \\ \downarrow \psi \\ \psi\left(f^{-1}s\right) \in \mathcal{A}(U) = (f_* A)(V) \end{array}$$

Now note that we effectively get the functoriality for free here since $f^{-1}f_*\mathcal{A} \to \mathcal{A}$, which is a morphism in $\mathbf{Sh}(X)$ corresponds to the identity $\mathrm{id} : f_*\mathcal{A} \to f_*\mathcal{A}$. The map $p_{\mathcal{A}} : f^{-1}f_*\mathcal{A} \to \mathcal{A}$ is the counit of the adjunction. Similarly the map $p_{\mathcal{B}} : \mathcal{B} \to f_* f^{-1}\mathcal{B}$ is the counit of $\mathrm{id} : f^{-1}\mathcal{A} \to f^{-1}\mathcal{B}$ then for $\varphi : B \to f_* A$

$$f^{-1}B \xrightarrow{f^{-1}\varphi} f^{-1}f_* A \xrightarrow{p_A} A$$

□

2. Morphisms of ringed spaces

Recall a ringed space is just a space with a sheaf of rings. Then a morphism of ringed spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a triple $\left(f, f^\flat, f^\#\right)$ where

$$f : X \to Y \qquad f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X \qquad f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

Note that $f^\flat$ determines $f^\#$ and vice versa, so we don't actually need to specify them separately.

Example 1. Suppose $\mathcal{O}_X \subseteq \mathrm{Fun}(X, k)$ and $\mathcal{O}_Y \subseteq \mathrm{Fun}(Y, k)$ are sheaves of functions. For a continuous map $f : X \to Y$, for any open set $V \subseteq Y$, $(U = f^{-1}V)$ then for any function $a \in \mathrm{Fun}(Y, k)(V)$,

$$a \circ f \in \mathrm{Fun}(X, k)(U) = f_* \mathrm{Fun}(X, k)(V)$$

so this is a map

$$f^\flat : \mathrm{Fun}(Y, k) \to f_* \mathrm{Fun}(X, k)$$

In particular, if $f^\flat(\mathcal{O}_Y) \subseteq f_*\mathcal{O}_X$, then we get $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$.