LECTURE 19
MATH 256A

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1. Local rings

Recall a local ring is a ring with a unique maximal ideal. Equivalently, every element outside of this ideal is invertible. Recall that for any ring and prime ideal $P \subseteq R$, R_P is a local ring with PR_P the unique maximal ideal.

If you have a homomorphism of local rings, you can put a condition on it that it sort of respects the local structure. A morphism

$$\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$$

is *local* iff $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. Equivalently $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$. This is equivalent since we always have $\varphi^{-1}(\mathfrak{n}) \subseteq \mathfrak{m}$.

Also recall that for any local ring $(R, \mathfrak{m})$ we have its residue field $\kappa_R = R/\mathfrak{m}$.

Example 1. We should think of this as continuous functions in a neighborhood of a point, modulo equality in that neighborhood. In particular, if we take $\mathcal{O}_X$ to be the sheaf of real-valued functions on any space X, then the stalk at any point consists of germs of functions and is a local ring with maximal ideal consisting of germs of functions with value 0 at the point.

Whenever we have a local homomorphism we also get a map on the residue fields since $\varphi : R \to S$ local induces a map $R/\mathfrak{m} \to S/\mathfrak{n}$ which is injective.

2. Locally ringed spaces

A locally ringed space $(X, \mathcal{O}_X)$ is a topological space and a sheaf of rings $\mathcal{O}_X$ such that for all $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

2.1. Examples.

Example 2. For X any space, we can consider continuous k-valued functions $\mathcal{C}(X, k)$ for some topological field k with non-trivial topology. So let $\mathcal{O}_X = \mathcal{C}(X, k)$, and then $\mathcal{O}_{X,p}$ consists of the continuous functions in a neighborhood of p, which is just k.

Example 3. We can also consider a sub-sheaf $\mathcal{O}_X \subseteq \mathcal{C}(X, k)$ as long as it is closed under taking inverses. I.e. for $f \in \mathcal{O}_X(U)$, we need it to be the case that for all $p \in U$, $f(p) \neq 0$ implies $f^{-1} \in \mathcal{O}_X(U)$.

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Example 4. We can consider X to be a classical variety over an algebraically closed field, and $\mathcal{O}_X$ to be the regular functions. Indeed if a polynomial function is nonzero at a point it is nonzero in a neighborhood since we defined the zero locus of it to be closed in X.

Example 5. For $X = \operatorname{Spec} R$ and $\mathcal{O}_X = \tilde{R}$, by definition $\mathcal{O}_{X,\mathfrak{p}} = R_{\mathfrak{p}}$ is local for any prime $\mathfrak{p}$.

2.2. **Morphisms.** We want a category **LRSp** of locally ringed spaces, so we need a notion of morphism. A morphism in **LRSp** is a ringed space morphism between these objects.

$$(X, \mathcal{O}_X) \xrightarrow{\left(f, f^\flat, f^\#\right)} (Y, \mathcal{O}_Y)$$

such that for all $p \in X$,

$$f_p^\# : \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p}$$

is a local homomorphism of local rings.

2.3. **Relationship with ring homomorphisms.** We want to see how Spec is a functor from $\mathbf{Cring}^{\mathrm{op}} \to \mathbf{LRSp}$. Given a homomorphism of rings $R \leftarrow S : \alpha$, we want to get a morphism $\left(\varphi, \varphi^\flat, \varphi^\#\right) : X \to Y$ where $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$.

For $P \in X$, define $\varphi(P) = \alpha^{-1}(P) = Q \in \operatorname{Spec} S = Y$. To see this is continuous, let $Z \subseteq Y$ be closed, so $Z = V(I)$ for some ideal I. But $P \in \varphi^{-1}(Z)$ iff $\varphi(P) \in Z$, iff $\alpha^{-1}(P) \supseteq I$ iff $P \supseteq \alpha(I)$, so $\varphi^{-1}(Z) = V(\alpha(I)) = V(J)$ where J is the ideal generated by $\alpha(I)$, so it is closed.

We define $\varphi^\#$ as follows. Let $P \in X$, and $Q = \varphi(P)$. We know $f^{-1}\mathcal{O}_{Y,P} = \mathcal{O}_{Y,Q} = S_Q$ and $\mathcal{O}_{X,P} = R_P$ so we want a map $S_Q \to R_P$. We have $\alpha : S \to R$, and then we can just compose with the localization map $R \to R_P$ to get a map $S \to R_P$. But then we have the following diagram:

$$\begin{array}{ccc} S_Q & \overset{\varphi_P^\#}{\dashrightarrow} & R_P \\ {\scriptstyle j_S}\uparrow & \nearrow & \uparrow{\scriptstyle j_R} \\ S & \xrightarrow{\varphi} & R \end{array}$$

We define $\varphi^\flat$ as follows. We want a sheaf homomorphism on Y $\mathcal{O}_Y \to \varphi_* \mathcal{O}_X$. In principal we should have to specify this for every open set, but the Sheaf axiom tells us that we actually only have to do it for a base. Recall that in Y, the open sets $Y_f = Y \setminus V(f)$ form a base of the topology. But now $Y_f \cap Y_g = Y_{fg}$, so the intersections are in the base already, so it suffices to specify:

$$\varphi^\flat(Y) : \varphi_* \mathcal{O}_X(Y_f) = \mathcal{O}_Y(Y_f) \to \mathcal{O}_X\left(\varphi^{-1} Y_f\right)$$

We know that:

$$\varphi^{-1}(V(f)) = V(\alpha(f)) \qquad \varphi^{-1}(Y_f) = X_{\alpha(f)}$$

and

$$\mathcal{O}_Y(Y_f) = S_f \qquad \mathcal{O}_X\left(\varphi^{-1} Y_f\right) = \mathcal{O}_X\left(X_{\alpha(f)}\right) = R_{\alpha(f)}$$

Now in analogy with the previous construction we have the following diagram commutes:

$$\begin{array}{ccc} S_f & \overset{\varphi^\flat(Y_f)}{\dashrightarrow} & R_{\alpha(f)} \\ \uparrow & \nearrow & \uparrow \\ S & \xrightarrow{\alpha} & R \end{array}$$

Exercise 1. Check that this is all actually functorial.