LECTURE 2
MATH 256

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1. General theory of affine varieties

Let $X = V(F)$ and $Y = V(G)$ be two varieties, then clearly $X \cap Y = V(F \cup G)$ is a variety. Somewhat less obvious is the fact that $X \cup Y = V(F \cdot G)$ is a variety, where $F \cdot G$ consists of all products of functions in F and functions in G.

In an informal sense, it is clear that k^n is n dimensional, and $X = V(f)$ will generically have dimension $n-1$. In some sense, more equations reduce the dimension. This will eventually all be worked out using a theory of dimension for Noetherian schemes, but for now we will keep the discussion informal.

Example 1. The plane itself has dimension 2, and the graph of a polynomial f is generically a one dimensional curves. Then insisting that two equations are satisfied, as long they intersect, we get a point. Indeed, to get any point (a, b) we can just take $V(x-a, y-b)$. This is of course not always true, since not all pairs of polynomials allow any simultaneous solutions.

Example 2. Note that every new equation does not always decrease the dimension by 1, for example if we have $V(xy)$, this is the xz plane and the yz plane. Then introducing xz, we get $V(xy, xz)$ which is the yz plane along with the x axis, which also has dimension 2.

Fact 1. *Introducing k polynomials to an n dimensional variety will be at least dimension $n-k$.*

We will now write a full list of affine varieties $X \subset k^1$ in the line. Of course we have $X = k = V(0)$ and $X = \emptyset = V(1)$. Besides these cases, the only other potential varieties are finite sets. For some polynomial f, $V(f)$ will always only consist of a finite set of points. Recall that the polynomials in one variable form a PID, and therefore a UFD, so we can factor any such polynomial f and just take the union of the varieties defined by such factors.

Now we will think about subvarieties $X \subset k^2$ of the plane. Of course we first have $X = k^2$ and $X = \emptyset$. Then for some polynomial f, consider $V(f)$. The polynomials in two variables no longer form a PID, but they are still a UFD, so we can factor f as

$$f = f_1(x,y)^{e_1} \cdots f_k(x,y)^{e_k}$$

and then as before,

$$V(f) = \bigcup_i V(f_i)$$

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So all we get are curves in the plane, and finite sets of points. On any curve, the vanishing locus of any function on that curve is either the whole curve, or some finite set of points.

2. Maps of varieties

Not only should we talk about the sorts of varieties we can get, but we should also talk about maps between varieties. A fundamental theme in algebraic geometry is considering families of varieties, and in particular that we can consider a family of varieties as a variety itself. Even though we haven't defined it, consider a map of varieties $f : X \to Y$. The preimages of points of Y will also be subvarieties of X, so we can parametrize these subvarieties using points in Y.

Maps also come up when considering algebraic groups. A lot of important groups are algebraic groups, which are just groups which also happen to be algebraic varieties. But there needs to be some sort of compatibility between these two structures in the sense that the group multiplication and inverse map are maps of algebraic varieties.

So let $X \subseteq k^n$ and $Y \subseteq k^m$ be two algebraic varieties. For $\phi : X \to Y$ to be a map of algebraic varieties, we want the coordinates for $y = \phi(x)$ to be given by $y_i = f_i(x_1, \cdots, x_n)$ for polynomials f_i.

We now consider some examples of algebraic groups.

Example 3. The group SL_n,

$$\mathrm{SL}_n \coloneqq \{A \in M_n \mid \det(A) = 1\}$$

is contained in $M_n = k^{n^2}$, and can be viewed as an algebraic variety $V(\det(X) - 1) = \mathrm{SL}_n$.

Then multiplication in SL_n is just matrix multiplication, and this is indeed given by a polynomial in the entries of the two matrices which of course maps the subvariety $\mathrm{SL}_n \times \mathrm{SL}_n \subseteq k^{2n^2}$ to the subvariety $\mathrm{SL}_n \subseteq k^{n^2}$. The inverse map is also a polynomial map from $\mathrm{SL}_n \to \mathrm{SL}_n$ so SL_n is an algebraic group.

All of the semi-simple Lie groups are algebraic, though this is trickier to see for some.

Example 4. Consider the group GL_n consisting of the invertible matrices $X \in M_n$. This just means $\det(X) \neq 0$, but this isn't enough to view GL_n as a variety in k^{n^2}, since it is not defined using polynomials, but rather by insisting that a polynomial happens to be nonzero. To do this, we need to introduce an extra coordinate to get coordinates $x_{11}, \cdots, x_{n,n}, z$. Now we can write

$$\mathrm{GL}_n = V(z \det(X) - 1)$$

so $\mathrm{GL}_n \subseteq k^{n^2+1}$ is an affine subvariety of k^{n^2+1}. One might wonder if it is just inconvenient to embed GL_n as an affine subvariety of k^{n^2} itself, or if it is in fact impossible. As it turns out, this is not possible, and $n^2 + 1$ is the minimal such dimension. We have a projection map $k^{n^2+1} \to k^{n^2}$, but this turns out to be an injective map when restricted to $\mathrm{GL}_n \subset k^{n^2+1}$ since only one z value can be associated to an invertible matrix.

Now consider the group multiplication $\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$. This is clearly an algebraic map in k^{n^2}, but we need to check this with the new coordinate. For two

$X, Y \in \mathrm{GL}_n$,
$$z = \frac{1}{\det(XY)} = \frac{1}{\det(X)} \frac{1}{\det(Y)}$$
so it turns out fine. The inverse map bringing $\mathrm{GL}_n \ni A \mapsto A^{-1} \in \mathrm{GL}_n$ also happens to be an algebraic map, even with the new coordinate.

Example 5. Now we consider an example of a non-affine variety. The simplest such example is the plane minus the origin $k^2 \setminus \{(0,0)\}$. Since $(0,0)$ is a subvariety, this object should be an open subset. And however we end up developing the theory, we certainly want open subsets of varieties to be varieties. This example cannot be globally defined by polynomials, but as in lecture 1, we only insisted that varieties be locally defined by polynomials.

We regard $U_x = k^2 \setminus V(x) \subset k^2$ as a variety by adding an extra coordinate v, and insist that $v = 1/x$, so $V = V(vx - 1) \simeq U_x$, and then similarly we can think of $U_y = k^2 \setminus V(y) \subset k^2$ as a variety by introducing the coordinate w and taking $W = V(wy - 1) \simeq U_y$.

Now the original open set is covered by two open subvarieties. But for this to make sense, this all needs to be compatible on $U_x \cap U_y$. We can again regard $U_{xy} = k^2 \setminus V(xy)$ as a variety by introducing the variable z, and considering $Z = V(zxy - 1) \simeq U_{xy}$.

Now since $z = (xy)^{-1}$, we map $Z \to V$ by $(z, x, y) \mapsto (zy, x, y)$ and $Z \to W$ according to $(z, x, y) \mapsto (zx, x, y)$. Now we can check everything is well defined and we are done.