LECTURE 20
MATH 256A

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1. Examples

Last time we saw that the relationship between rings R and the objects of **LRSp** given by $\operatorname{Spec} R$ is functorial in the sense that a morphism of rings f gives rise to a morphism (in the opposite direction) $\left(f, f^{\#}, f^{\flat}\right)$. We will now see some examples which tell us what these morphisms look like.

Example 1. If we begin with a morphism $f : X \to Y$ of classical affine varieties. We attach two rings to this picture, $R = \mathcal{O}(X)$ and $S = \mathcal{O}(Y)$ where these consist of the regular functions on these spaces. Recall that if we take R to be the polynomials, then $\operatorname{Spec} R$ will effectively be X, only we get a few extra points.

In this picture, f yields a ring morphism $\alpha : S \to R$ which just maps $g \mapsto g \circ f$. From this we want to get a map of schemes

$$\tilde{X} = \operatorname{Spec} R \to \tilde{Y} = \operatorname{Spec} S$$

Recall that X can be identified with the closed (or classical points) $\tilde{X}_{\mathrm{cl}}$, and similarly $Y = \tilde{Y}_{\mathrm{cl}}$. More generally, $\tilde{X}$ consists of the irreducible closed subvarieties of X since $Z \leftrightarrow \mathcal{I}(Z) \subseteq R$, and the irreducible ones are points in $\operatorname{Spec} R$.

Consider the map taking $P \in \operatorname{Spec} R$ to $\alpha^{-1} P \in \operatorname{Spec} S$. This is just:

$$\alpha^{-1} P = \{ g \in S \mid g \circ f \in \mathcal{I}(Z) \} = \mathcal{I}\left(\overline{f(Z)}\right) = Q \subseteq S$$

so it is at least reasonable on points since it reduces to f on $X = \tilde{X}_{\mathrm{cl}}$.

Now what does this do to functions? In the situation of a morphism of affine schemes like this, it is easier to first think of the flat version:

$$f_{\alpha}^{\flat} : \mathcal{O}_{\tilde{Y}} \to f_* \mathcal{O}_{\tilde{X}} \ .$$

This is determined by what it does to open sets of the form $\mathcal{O}_{\tilde{Y}}\left(\tilde{Y}_g\right) = S_g$. So send this to $f_* \mathcal{O}_{\tilde{X}}\left(\tilde{Y}_g\right)$, but this is by definition just

$$f_* \mathcal{O}_{\tilde{X}}\left(\tilde{Y}_g\right) = \mathcal{O}_{\tilde{X}}\left(f^{-1} \tilde{Y}_g\right) = \mathcal{O}_{\tilde{X}}\left(\tilde{X}_{\alpha(g)}\right) = R_{\alpha(g)}$$

In particular, for $g = 1$, $f_{\alpha}^{\flat}$ is just α, and in the classical case this reduces to:

$$f_{\alpha}^{\flat} : \mathcal{O}_{\tilde{Y}}(U) = \mathcal{O}_Y(U_{\mathrm{cl}}) \to f_{\alpha,*}(U) = \mathcal{O}_{\tilde{X}}\left(f_{\alpha}^{-1} U\right) = \mathcal{O}_X\left(f^{-1} U_{\mathrm{cl}}\right)$$

which is literally just $g \mapsto g \circ f$.

For $x = \operatorname{Spec} R$, recall that we have the following basic theorem concerning the structure sheaf of a scheme:

Date: October 8, 2018.

Theorem 1. $\mathcal{O}_X(X) = R$

And it's not trivial, but there is also the following corollary:

Corollary 1. *$\mathcal{O}_X(X) = \mathcal{O}(X)$ for a classical variety.*

Now we want to consider some examples which consider an arbitrary ring, and then have a special kind of morphism.

Example 2. Consider $\alpha : R \to R_f$ for any R and $f \in R$. Then this gives us a map $j : \operatorname{Spec} R_f \to \operatorname{Spec} R$. If we write $X = \operatorname{Spec} R$ we have a natural correspondence (at least on the level of sets) $X_f = \operatorname{Spec} R_f$. Now we want to construct $\left(j, j^\flat, j^\#\right)$. In this case $j^\#$ is probably easier since it maps:

$$j^\# : j^{-1}\mathcal{O}_X = \mathrm{O}_X\,|_{X_f} \to \mathcal{O}_{X_f}$$

But in general, the inverse image is trivial in the case of an open embedding. Therefore at a point $P \in X_f$ we have:

$$j^\# : R_P \to R_{f,P}$$

but these two rings are isomorphic, so this construction works as we would expect in this case as well.

This was basically just unravelling definitions, but we also know that

$$j^\flat : \mathcal{O}_X(X) \to j_*\mathcal{O}_{X_f}(X_f) = \mathcal{O}_{X_f}(X_f)$$

is just a map $R \to R_f$. It is a general fact that:

Fact 1. *Any morphism of affine schemes $f : X = \operatorname{Spec} R \to Y = \operatorname{Spec} S$ is determined by the corresponding ring homomorphism*

$$f^\flat : \mathcal{O}_Y(Y) = S \to \mathcal{O}_X(X) = R$$

But what is secretly happening here is much stronger:

Theorem 2. *For $Y = \operatorname{Spec} S$ and $X \in$ **LRSp**:*

$$X \to Y = \operatorname{Spec} S$$

is equivalent to

$$\mathcal{O}_X(X) \leftarrow S$$

so it turns out X doesn't have to be an affine scheme. This is probably the most important theorem in the whole subject. We will prove it soon, but we mention it now since this example motivates it.

This essentially means that the trivial functor sending a locally ringed space to its ring of local functions is adjoint to Spec, but adjoint functors are unique, so all of this Spec business was forced upon us.

Besides motivating the theorem, the point of the previous example was to see that even though, in general, it is hard to find the inverse image sheaf, it is easy for open embeddings. It is also hard in general to find stalks, but this is easy for closed immersions as we will see in the following example.

Example 3. Consider a closed immersion i.e. an inclusion of a closed subvarieties. Start with a surjective ring homomorphism $R \twoheadrightarrow R/I$. Then let $X = \operatorname{Spec} R$ and $Z = \operatorname{Spec}(R/I)$. Then we get a morphism $Z \to X$, and the question is what this morphism looks like. Of course ideals of R/I correspond to ideals of R containing

I, so we can do this for primes too, but this just means $\operatorname{Spec}(R/I) = V(I)$, so this map $Z \to X$ is just the natural inclusion.

Let's look at stalks.

$$\left(i^{-1}\mathcal{O}_X\right)_P = \mathcal{O}_{X,P} = R_P$$

and then $i_P^{\#}$ maps to:

$$\mathcal{O}_{Z,P} = (R/I)_P = R_P/I_P$$

so for $P \in Z$ there is a unique map $i_P^{\#} : \mathcal{O}_{X,P} \to \mathcal{O}_{Z,P}$ or just $i_P^{\#} : R_P \to R_P/I_P$ which is surjective. Then the stalks are

$$i_*\mathcal{O}_Z = \mathcal{O}_X/\tilde{I}$$

so the homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is just quotienting by $\tilde{I}$. But what exactly does this quotient mean? Taking the obvious quotient (quotient in each open set) will not always be a sheaf, so we have to sheafify.