LECTURE 21
MATH 256A

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We could delay this discussion, or we also could have had it earlier, but we will have it now.

1. Sheafification

1.1. **Definition.** Let $\mathcal{A}$ be a presheaf on X. Define the stalk the same way as we did for a sheaf:

$$\mathcal{A}_p = \varinjlim_{U \ni p} \mathcal{A}(U)$$

More explicitly there is a germ map:

$$\mathcal{A}(U) \to \mathcal{A}_P$$

which maps $a \mapsto a_P$ to the germ. One thing that changes from the sheaf case, is that the map

$$\mathcal{A}(U) \to \prod_{P \in U} \mathcal{A}_P$$

isn't injective since this fact used the sheaf axiom.

Now observe that there is a sheaf:

$$U \mapsto \prod_{P \in U} \mathcal{A}_P$$

which has a subsheaf called the sheafification $\mathcal{A}^{\mathrm{Sh}}$ which consists of the elements locally given by sections of $\mathcal{A}$. We can explicitly send

$$\mathcal{A} \to \mathcal{A}^{\mathrm{Sh}}$$

where $a \in \mathcal{A}(U)$ maps to $(a_P)_{P \in U}$.

Remark 1. Recall that for $f : X \to Y$ a map of topological spaces, we could turn a sheaf $\mathcal{S}$ on Y into a sheaf $f^{-1}\mathcal{S}$ on X with the inverse image functor. In particular, we saw that even if $\mathcal{S}$ was only a presheaf we would end up with a sheaf. As it turns out, for $X = Y$, and f the identity, this construction is exactly sheafification. this motivates the alternative notation (use by Grothendieck):

$$\mathcal{A}^{\mathrm{Sh}} = 1_X^{-1} \mathcal{A}$$

As a first property of the sheafification we might notice that the stalks agree with the original presheaf: $\mathcal{A}_P \simeq \left(\mathcal{A}^{\mathrm{Sh}}\right)_P$. We also know that in general

$$\operatorname{Hom}\left(f^{-1}\mathcal{A}, \mathcal{B}\right) = \operatorname{Hom}\left(\mathcal{A}, f_* \mathcal{B}\right)$$

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so if $\mathcal{B} \in \mathbf{Sh}(X)$ and $\mathcal{A} \in \mathbf{PreSh}(X)$, then we have a map:

$$\begin{array}{c} \operatorname{Hom}(\mathcal{A}, \mathcal{B}) \\ \uparrow \\ \operatorname{Hom}\left(\mathcal{A}^{\mathrm{Sh}}, \mathcal{B}\right) \end{array}$$

given by composition. Note that if $\mathcal{B}$ is a sheaf, this is an isomorphism. This is all saying that $(-)^{\mathrm{Sh}}$ is left adjoint to the functor:

$$\mathbf{Sh}(X) \to \mathbf{PreSh}(X)$$

We also have the universal property

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & \nearrow & \\ \mathcal{A}^{\mathrm{Sh}} & & \end{array}$$

for $\mathcal{B}$ a sheaf.

2. Quotient sheaves

2.1. **Motivation.** Recall last lecture we considered a surjective ring homomorphism $R \to R/I$ which corresponded to a morphism $\operatorname{Spec} R/I \to R$ and in turn gave us maps $i_P^{\#}$ on stalks given by the quotient. This motivates us to define the quotient of two sheaves. However the issue is that the naive definition of this isn't always well-behaved. In particular, the "pre-sheaf-quotient" of a sheaf by a sheaf isn't always a sheaf.

Example 1. Let X be the unit circle, and consider the sheaf $\mathcal{C} = \mathcal{C}(X, \mathbb{R})$ of continuous real-valued functions on X. Also consider $\mathbb{Z}$ the sheaf of locally constant $\mathbb{Z}$-valued functions. This is really $\mathcal{C}(X, \mathbb{Z})$ for $\mathbb{Z}$ equipped with the discrete topology. Then the map $\mathbb{Z} \xrightarrow{2\pi\cdot} \mathcal{C}$ gives us a subsheaf of $2\pi\mathbb{Z}$-valued functions.

An initial guess for the sheaf quotient might be just the natural quotient of presheaves:

$$U \to \mathcal{C}(U) /_{\mathrm{pre}} \mathbb{Z}(U)$$

There is an element $\theta_U \in \mathcal{C}(U) / \mathbb{Z}(U)$ for every open set $U \subsetneq X$ where we just pick a point not in U, and then take the function which measures the angle such that this point we picked is the branch cut. What is certainly true, is that θ_U and θ_V will always agree on $U \cap V$ as an element of the quotient. So if the sheaf axiom is to hold on this quotient, there would have to be some $\theta \in \mathcal{C}(X) /_{\mathrm{pre}} \mathbb{Z}(X)$, but we can't represent the angle function as any continuous real-valued function on the circle since it would always have a discontinuity.

It's easy fix this by taking the sheafification. So we define:

$$\mathcal{C}/\mathbb{Z} := \left(\mathcal{C}/_{\mathrm{pre}} \mathbb{Z}\right)^{\mathrm{Sh}} = \mathcal{C}(X, \mathbb{R}/2n\mathbb{Z})$$

Another way to express this is that we have a sort of exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi\cdot} & \mathcal{C} & \longrightarrow & \mathcal{C}/\mathbb{Z} & \longrightarrow & 0 \\ & & & & & \searrow \quad \nearrow & & & \\ & & & & & \mathcal{C}/_{\mathrm{pre}} \mathbb{Z} & & & \end{array}$$

Now take global sections $\Gamma(X, -)$ which just sends $\mathcal{S} \to \mathcal{S}(X)$. This gives us another sequence:

$$0 \to \mathbb{Z}(X) = \mathbb{Z} \to \mathcal{C}(X) \to \mathcal{C}/\mathbb{Z}(X)$$

However the last map is not exact. So Γ is still left exact, but not exact. The global section functor can be thought of as the zeroth object in a sequence of higher cohomology functors.

Whenever we have an exact sequence of sheaves we get

$$0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathcal{C}) \to H^0(X, \mathcal{C}/\mathbb{Z}) \to H^1(X, \mathbb{Z})$$

and in this case of the unit circle, the point is that it isn't simply connected. The generic idea here is that sheaf cohomology is somehow detecting the ordinary homology of the space.

2.2. **General discussion.** More generally consider a morphism of sheaves $f : \mathcal{M} \to \mathcal{N}$. Now we want to define $\ker f$ as

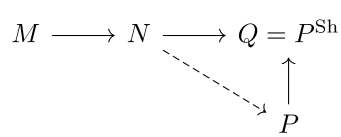
$$K = \ker f := \ker_{\mathbf{PreSh}}(f)$$

where $U \mapsto \ker(\mathcal{M}(U) \to \mathcal{N}(U))$. This is a sheaf, and clearly satisfies the usual universal property defining a kernel.

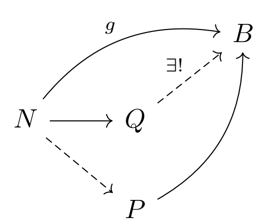
If we define $\operatorname{coker} f$ the same way, this won't be a sheaf in general. So instead we define:

$$Q = \operatorname{coker} f := \operatorname{coker}_{\mathbf{PreSh}}(f)^{\mathrm{Sh}} = P^{\mathrm{Sh}}$$

The picture is:



and then by the universal property, we get



where $g \circ f = 0$, so indeed this is a cokernel in the universal sense.

So we have an exact sequence $0 \to K \to M \to N \to Q \to 0$, and we can look at the stalks,

$$0 \to K_P \to M_P \to N_P \to Q_P \to 0 \tag{1}$$

but sheafification doesn't change the stalks, so $Q_P = P_p$ so we claim that this is in fact an exact sequence.

Exercise 1. Show the sequence (1) is exact.

2.3. **Abelian categories.**

Definition 1 (Additive category)**.** A category $\mathbf{A}$ is an additive category iff:

(1) Every Hom set has the structure of an abelian group in such a way that composition distributes over addition.
(2) $\mathbf{A}$ has a zero object.
(3) $\mathbf{A}$ has a product $X \times Y$ for any two objects $X, Y \in \mathrm{Obj}(\mathbf{A})$.

Definition 2 (Additive functor)**.** A functor between additive categories is an *additive functor* iff it induces a group homomorphisms on the Hom sets.

Definition 3 (Kernel, cokernel)**.** Let f be any morphism in an additive category $\mathbf{C}$. The *kernel* of f is a morphism i which is universal with respect to the property that $fi = 0$. The *cokernel* of f is a morphism e which is universal with respect to the property that $ef = 0$.

Definition 4 (Monomorphism, epimorphism)**.** In any category $\mathbf{A}$, a morphism i is a monomorphism iff

$$ig = ih \implies g = h$$

for any two morphisms g, h. A morphism e is an epimorphism iff

$$ge = he \implies g = h$$

for any two morphisms g, h.

Definition 5 (Abelian category)**.** An additive category $\mathbf{A}$ is an *abelian category* iff

(1) Every morphism has a kernel and cokernel.
(2) Every monomorphism in $\mathbf{A}$ is the kernel of its cokernel.
(3) Every epimorphism in $\mathbf{A}$ is the cokernel of its kernel.

Theorem 1. *The category* $\mathbf{Sh}(X)$ *is an abelian category.*

Proof. This is effectively a consequence of the fact that stalks detect everything, and these are in an abelian category. □