LECTURE 22
MATH 256A

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Today we will talk about closed subschemes. We've already seen the main idea of this sort of vaguely, but now we know exactly what that actually means.

1. Closed embeddings of ringed spaces

1.1. **General case.** Consider a ringed space $(X, \mathcal{O}_X)$ and a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then we can look at $\mathcal{O}_X/\mathcal{I}$, and consider X with this as a sheaf of rings. But this is somehow annoying since there will be points where this is locally zero, so it isn't even a locally ringed space since the zero ring is not local. So we insist this has nonzero stalks. But $(\mathcal{O}_X/\mathcal{I})_p = 0$ is equivalent to $1_p = 0$ i.e.

$$\left\{p \mid (\mathcal{O}_X/\mathcal{I})_p \neq 0\right\} = \mathrm{Supp}\,(1_P)$$

which is closed.

Another nice thing that happens is that for any closed embedding

$$i : Z \hookrightarrow X$$

there is a natural bijection between all sheaves of Z and sheaves in X supported in Z:

$$Z \xhookrightarrow{\qquad\qquad i \qquad\qquad} X$$

$$\mathbf{Sh}\,(Z) \underset{i^{-1}}{\overset{i_*}{\rightleftarrows}} \{\mathcal{M} \in \mathbf{Sh}\,(X) \mid \mathrm{Supp}\,(\mathcal{M}) \subseteq Z\}$$

$$\left(\mathcal{N} \leftarrow i^{-1} i_* \mathcal{N}\right) \qquad\qquad \left(\mathcal{M} \to i_* i^{-1} \mathcal{M}\right)$$

and in particular,

$$(i_*\mathcal{N})_P = \begin{cases} 0 & p \notin Z \\ \mathcal{N}_P & p \in Z \end{cases}$$

so both functors preserve stalks in this case, and therefore these are isomorphisms and this is an equivalence of categories.

From this it follows that there is a unique $\mathcal{O}_Z$ such that $\mathcal{O}_X/\mathcal{I} = i_*\,(\mathcal{O}_Z)$, and in particular $\mathcal{O}_X = i^{-1}\mathcal{O}_X/\mathcal{I}$. So we get a morphism of ringed spaces:

$$\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$$

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1.2. **For locally ringed spaces.** This worked for any ringed space, but if we are dealing with locally ringed spaces we can say a bit more. So let X be locally ringed. Then $(\mathcal{O}_X/\mathcal{I})_p \neq 0$ iff $\mathcal{I}_p \subseteq \mathfrak{m}_p$. The vanishing locus is exactly

$$Z = V(\mathcal{I}) = \{p \mid \mathcal{I}_p \subseteq \mathfrak{m}_p\}$$

In particular, for $X = \operatorname{Spec} R$ and $\mathcal{I} = \tilde{I}$, then $\operatorname{Spec} R/I$ as a set is the primes containing I, and since this is the support of Z, $\operatorname{Spec} R/I = Z$, and the associated homomorphism of sheaves of rings at points is

$$R_p \mapsto (R/I)_p = R_p/I_p$$

so this says

$$\mathcal{O}_X \to \mathcal{O}_X/\tilde{I}$$

Remark 1. When $X = \operatorname{Spec} R$, we might wonder whether or not $\mathcal{I} \subseteq \mathcal{O}_X$ must come from $I \subseteq R$. This is not the case, but it will be true iff $\mathcal{I}$ is a quasi-coherent sheaf of ideals. This has a definition that makes sense in any ringed space, but we won't talk about it now. This theorem is called QCO, which we will sort of just take for granted for now.

This means that for X a scheme, and $\mathcal{I} \subseteq \mathcal{O}_X$ a quasi-coherent ideal sheaf, we can consider $i : Z \to X$, and cover X with affines $U = \operatorname{Spec} R$ and since $\mathcal{I}$ is quasi-coherent, it will come from an ideal in that ring, so $Z \cap U = \operatorname{Spec} R/I$ will embed in U. This isn't obvious right now, but quasi-coherent ideal sheaves correspond exactly with such embeddings.

2. Radical ideals

If $Z \subseteq X$ is a closed subset of a scheme, then for $\operatorname{Spec} R = U \subseteq X$ we can consider $I = \mathcal{I}(Z \cap U)$. We know $\sqrt{I} = I$ is the largest ideal such that $V(I)$ is Z. So this will give us an ideal sheaf $\mathcal{I}(Z)$, which is quasi-coherent, and locally of the form $\tilde{I}$ for $I = \mathcal{I}(Z \cap U)$ on affines U. This will be characterized by having

$$\mathcal{O}_Z = i^{-1}\mathcal{O}_X/\mathcal{I}(Z)$$

The ring R/I is reduced, which means the only nilpotent element is 0. In general a ring S is reduced iff S_P is reduced fr all $P \in \operatorname{Spec} S$. So $\mathcal{O}_Z$ is a sheaf of reduced rings. So from a closed subset, we get a unique "largest" quasi-coherent ideal sheaf that has Z as its vanishing locus. I.e. Z comes with a unique reduced embedded closed subscheme structure.

Example 1. Consider the affine plane k^2. Intersect the x-axis, $V(y)$, with a parabola, $V\left(y - x^2 + a\right)$. So the intersections are $x = \pm\sqrt{a}$. So we think about $\operatorname{Spec} R = k^2$ (with bonus points). In the language of embeddings, we send

$$k[x, y] \to k[x, y]/(y) = k[x]$$

to get the x-axis, and we send

$$k[x, y] \to k[x, y]/\left(y - x^2 + a\right) \simeq k[x]$$

to get the parabola. Now we want to think about the intersection of these closed subschemes scheme theoretically, but this is just the largest closed subscheme contained in both, which is just taking the sum of the radical ideal sheaves, which isn't always radical but is always quasi-coherent. We can write:

$$\begin{aligned} k[x,y] \,/\, \big((y) + \big(y - x^2 + a\big)\big) &= k[x,y] \,/\, \big(y, y - x^2 + a\big) \\ &\cong k[x] \,/\, \big(x^2 - a\big) \\ &= k[x] \,/\, \big(x - \sqrt{a}\,\big(x + \sqrt{a}\big)\big) \end{aligned}$$

where a has a square root since $k = \overline{k}$. Now if we send

$$k[x] \to k \times k$$

by mapping $f(x) \mapsto f(\sqrt{a}, \sqrt{a})$, this is just $k \times k$. But if $a = 0$, this becomes $k[x] \,/\, \big(x^2\big)$, which is not reduced since x is a nonzero nilpotent element. Scheme theoretically, the intersection of a line and a parabola that touches at one point intersect to give us $k[x] \,/\, \big(x^2\big)$. This is a local ring with maximal ideal (x). We know $k[x] \,/\, (x) = k$, has space $\operatorname{Spec} k = \mathrm{pt}$, and similarly $\operatorname{Spec} k[x] \,/\, \big(x^2\big) = \mathrm{pt}$, This is a zero-dimensional scheme of length two. When $a \neq 0$, this has two points. So the scheme $\operatorname{Spec} k[x] \,/\, \big(x^2\big)$ is somehow the model for tangent spaces.