LECTURE 23
MATH 256A

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1. Ways to make schemes out of other schemes

1.1. **Closed subschemes.** Last time we discussed closed subschemes. I.e. for a ringed space X and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Then $Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is closed in X and $i_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}$ and $i^\flat : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ is surjective.

We also saw that if X is a locally ringed space then so is Z, and i is in **LRSp** by construction. That way

$$Z = V(\mathcal{I}) = \{p \,|\, \mathcal{I}_p \subseteq m_p\}$$

In particular if X is a scheme, and Z turns out to be a scheme, then this is a morphism of schemes. But Z won't always be a scheme. We will discuss this later, but roughly speaking if X is a scheme, we will have that $\mathcal{I}$ is locally $\tilde{I}$ for $I \subseteq R$ on $U = \operatorname{Spec} R \subseteq X$ iff $\mathcal{I}$ is quasicoherent. We haven't defined this yet, but if we take this for granted, as long as $\mathcal{I}$ is quasicoherent, we have $Z \cap U = \operatorname{Spec} R/I$, and the inclusion i into U will be the closed embedding of schemes corresponding to $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$. So i is an affine morphism. Z is a closed subscheme.

1.2. **Open subschemes.** Let $U \subseteq X$ be an open subset of a subscheme. If we take affines $W \subseteq X$ which cover X, then $W \cap U$ will be open in U. But these themselves are covered by W_fs which are also affine since they're just $\operatorname{Spec} R_f$, so U is a scheme.

1.3. **Locally closed subscheme.** This is somehow a combination of the last two. Let X be a topological space with subset $Z \subseteq X$. This is called *locally closed* if it is the intersection of an open subset with a closed subset. The typical example of this is somehow a curve with a hole.

Exercise 1. Show that this is equivalent to: for all $p \in Z$, there exists an open neighborhood $p \in U \subseteq X$ such that $Z \cap U$ is closed in U.

Any locally closed embedding can be written

$$Z \xhookrightarrow[\text{cl}]{i} U \xhookrightarrow[\text{op}]{j} X$$

which is equivalent to

$$Z \xhookrightarrow{j} Y \xhookrightarrow{i} X$$

It isn't quite like this for schemes. We have the implication:

$$Z \xhookrightarrow{j} Y \xhookrightarrow{i} X \qquad \Longleftarrow \qquad Z \xhookrightarrow{i} U \xhookrightarrow{j} X$$

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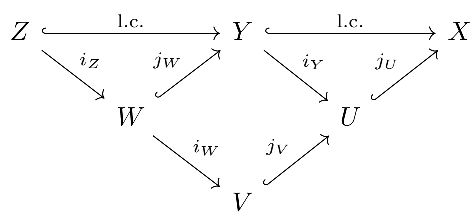
But this is not reversible, but for this to be the case, Z will have to have a non-reduced structure, and X will somehow not be quasi-compact.

Exercise 2. Find such an example.

Definition 1. If this actually is reversible, we say this is a locally closed subscheme/embedding.

Remark 1. Equivalently, a closed subscheme is a pair (Z, U) where Z is a locally closed subset and U is an open subscheme and $\overline{Z} \cup U = X$. In particular, we can take $U = X \setminus Z_b$ where $Z_b = \overline{Z} \setminus Z$, so specifying the pair is unnecessary. The point is we want to allow $Z \hookrightarrow X$ to factor as an open embedding followed by a closed embedding as long as it factors as a closed embedding followed by an open embedding.

Remark 2. In EGA this is just called a subscheme.



Example 1. For a collection of schemes $\{X_\alpha\}$, and we take

$$\coprod X_\alpha$$

as topological spaces, then this is a scheme basically since $\operatorname{Spec} R \times S = \operatorname{Spec} R \amalg \operatorname{Spec} S$. This is the coproduct in **LRSp** and therefore in the category of schemes as well. Equivalently, $\operatorname{Hom}(X, -)$ is in functorial bijection with

$$\prod_\alpha \operatorname{Hom}(X_\alpha, -)$$

1.4. **Gluing schemes.** Let's start out with just gluing sets. So suppose we have a set:

$$X = \bigcup_\alpha X_\alpha$$

but suppose all we know is X_α and want to describe X itself. So we need to describe how the X_α overlap with each other. I.e. for each pair α, β we're supposed to be given $X_{\alpha,\beta} \subseteq X_\alpha$, $X_{\beta,\alpha} \subseteq X_\beta$, and a bijection

$$\gamma_{\beta,\alpha} : X_{\alpha,\beta} \to X_{\beta,\alpha} .$$

We require:

1. $X_{\alpha,\alpha} = X_\alpha$ and $\gamma_{\alpha,\alpha} = \mathrm{id}_{X_\alpha}$
2. $\gamma_{\alpha,\beta} = \gamma_{\beta,\alpha}^{-1}$
3. $\gamma_{\theta,\beta} \circ \gamma_{\beta,\alpha} = \gamma_{\theta,\alpha}$ on $\gamma_{\beta,\alpha}^{-1}(X_{\beta,\theta})$ which must be contained in $X_{\alpha,\theta}$.

Then we can mod out the disjoint union by $\amalg X_\alpha / \sim$ where $X_\alpha \ni x \sim y \in X_\beta$ iff $y = \gamma_{\beta,\alpha}(x)$. Now we just have to check some details, but it does really work.

Now suppose the sets have extra structure. So let the X_α be topological spaces with the gluing data subject to the same axioms, only now we also require $X_{\alpha,\beta}$ is open in X_α and

$$\gamma_{\beta,\alpha} : X_{\alpha,\beta} \xrightarrow{\cong} X_{\beta,\alpha}$$

is a homeomorphism.

Now let $\mathcal{O}_\alpha$ be a sheaf on each X_α. Now the gluing is subject to the same conditions, only now we need:

$$\gamma_{\beta,\alpha} : \mathcal{O}_\beta|_{X_\alpha \cap X_\beta} \to \mathcal{O}_\alpha|_{X_\alpha \cap X_\beta}$$

isomorphisms. Then

$$\gamma_{\theta,\beta} \circ \gamma_{\beta\alpha} = \gamma_{\theta,\alpha}$$

on $X_\alpha \cap X_\beta \cap X_\theta$ which implies there is a unique sheaf $\mathcal{O}$ such that $\mathcal{O}|_{X_\alpha} = \mathcal{O}_\alpha$ and

$$\mathcal{O}_p = \mathcal{O}_{\alpha,p}$$

for $p \in X_\alpha$.