LECTURE 24
MATH 256A

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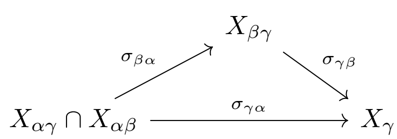
1. Gluing schemes

Recall last time we talked about pasting a collection of schemes X_α into one big scheme X. To do this, we have to specify the overlaps $X_{\alpha,\beta}$ between any pair X_α and X_β, and an explicit isomorphism $\sigma_{\beta\alpha} : X_{\alpha\beta} \to X_{\beta\alpha}$. Then these need to satisfy the so-called cocycle condition, which says the following diagram commutes:



Example 1. We can build a space $\mathbb{P}^n_A$ which will qualify as a projective over a cring A. For motivation, recall that we thought of

$$\mathbb{P}^n(k) = \{(x_0 : \cdots : x_n)\}$$

(for x_i not all 0) as $n-1$ dimensional subspaces of n-dimensional space. We saw this as a variety by covering it with pieces where one of the $x_i \neq 0$ and then the others are somehow coordinates, and this turns out to be isomorphic to k^n. But this somehow came glued together already, and we just had to be sure we were matching data on the pieces.

In the world of schemes, we will abstractly glue copies of affine n-space $\mathbb{A}^n(k) = k^n$. This should be thought of as $\operatorname{Spec} k[x_1, \cdots, x_n]$. In general we will take:

$$\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \cdots, x_n]$$

so this qualifies as affine n-space. Now consider a collection of schemes:

$$U_i = \mathbb{A}^n_A = \operatorname{Spec} A\left[x_{0/i}, \cdots, x_{n/i}\right] / \left(x_{i/i} - 1\right)$$

for $i \in \{0, \cdots, n\}$ and now we need to decide how to glue them. Specify the overlaps to be:

$$U_{i,j} = (U_i)_{x_{j/i}} = \operatorname{Spec} A\left[x_{0/i}, \cdots, x_{n/i}, x_{j/i}^{-1}\right] / \left(x_{i/i} - 1\right)$$

Now we need to specify the isomorphisms between the overlaps. We are lucky because the $U_{i,j}$ are affine, so since we that given a ring homomorphism we get a morphism between the associated affine scheme,[1] we just need to specify the ring

Date: October 17, 2018.

[1]In fact every morphism of affine schemes comes from a ring homomorphism, which we haven't proven but will soon.

homomorphisms:

$$\begin{array}{cc} A\left[x_{0/j},\cdots,x_{n/j},x_{i/j}^{-1}\right]/\left(x_{j/j}-1\right) & x_{k/j} \\ \downarrow & \downarrow \\ A\left[x_{0/i},\cdots,x_{n/i},x_{j/i}^{-1}\right]/\left(x_{i/i}-1\right) & x_{k/i}x_{j/i}^{-1} \end{array}$$

where this restricts to the identity on A. To check this is a ring homomorphism we need to check that $x_{i/j}$ maps to something invertible, and indeed it maps to $x_{i/j}\mapsto x_{i/i}x_{j/i}^{-1}=x_{j/i}^{-1}$ which is invertible and similarly, $x_{i/j}^{-1}\mapsto x_{j/i}$. Also $x_{k/j}x_{i/j}^{-1}\mapsto x_{k/i}$.

To see the cocycle condition we need the following to commute:

$$\begin{array}{ccc} & U_{j,k} & \\ \nearrow^{\sigma_{j,i}} & & \searrow^{\sigma_{k,j}} \\ U_{ij}\cap U_{ik} & \xrightarrow{\sigma_{k,i}} & U_k \end{array}$$

In the bottom direction:

$$x_{m/k}\mapsto x_{m/i}x_{k/i}^{-1}$$

and in the top direction

$$x_{m/k}\mapsto x_{m/j}x_{k/j}^{-1}\mapsto x_{m/i}x_{k/i}^{-1}\left(x_{j/i}x_{j/i}^{-1}\right)^{-1}=x_{m/i}x_{k/i}^{-1}$$

as desired.

As it turns out,

$$\mathbb{P}_A^n=\operatorname{Spec}(A)\times\mathbb{P}_{\mathbb{Z}}^n$$

we can call this the fiber product over $\operatorname{Spec}\mathbb{Z}$, but in fact since we have a unique map $\mathbb{Z}\to R$ we get a unique map $\operatorname{Spec}R\to\operatorname{Spec}\mathbb{Z}$, and if every affine scheme has a unique morphism to $\operatorname{Spec}\mathbb{Z}$ then every scheme does.

But if we have a map $\operatorname{Spec}R\to\operatorname{Spec}A$, this means we have a map $A\to R$ which is just saying R is an A-algebras, so a scheme over $\operatorname{Spec}A$ has structure sheaf which is in fact a sheaf of A-algebras.

So let T be a scheme over A i.e. it comes with a morphism $p:T\to\operatorname{Spec}A$, and then let $\psi:T\to\mathbb{P}_{\mathbb{Z}}^n$. But we have a morphism $\mathbb{P}_A^n\to\mathbb{P}_{\mathbb{Z}}^n$, and in particular the following is a commutative diagram of schemes over A:

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & \mathbb{P}_A^n \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}A & \end{array}$$

so $\mathbb{P}_A^n$ represents a functor on schemes over $\operatorname{Spec}A$ in the same way that $\mathbb{P}_{\mathbb{Z}}^n$ represents a functor on all schemes.