LECTURE 25
MATH 256A

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A general theme in geometry is that we want to think of schemes as the functor they represent. In particular, if we know the functor, we know the scheme.

1. Projective space

Recall the Grassmannian is classically just k dimensional subspaces in n dimensional space. We can view this as a variety with Plücker coordinates, but we might wonder if this somehow a natural/unique way to see this. Scheme theory will tell us that it is. We consider projective space as an example of this.

1.1. **General case.**

Recall last time we saw that a scheme T over $\operatorname{Spec} A$ i.e. a scheme where the structure sheaf is a sheaf of A algebras rather than just being a sheaf of rings fits in the following diagram:

$$\begin{array}{ccc} T & \longrightarrow & \mathbb{P}^n_A \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

and this is equivalent to

$$T \to \mathbb{P}^n_{\mathbb{Z}}$$

First we need to know what it means to give a morphism of schemes $T \to \operatorname{Spec} R$. We will put the proof of the following theorem on the to-do list but assume we know it for now:

Theorem 1. *Morphisms of schemes $T \to \operatorname{Spec} R$ from any scheme to an affine scheme correspond one-to-one to ring homomorphisms $R \to \mathcal{O}_T(T)$.*

Assume we have this map $\varphi : T \to \mathbb{P}^n_A$. Then we have the open sets $W_i = \varphi^{-1}(U_i)$ cover T. Recall the U_i are Spec of the ring

$$A\left[x_{0/i}, \cdots, x_{n/i}\right] / \left(x_{i/i} - 1\right)$$

now we want this to map to $\mathcal{O}_T(W_i)$. To specify this data we just need to specify it on the variables. So send the $x_{j/i} \mapsto f_{j/i}$ where the $f_{j/i}$ are basically arbitrary, except we insist $f_{i/i} = 1$. Now we need to make sure they're compatible on T, so on $W_i \cap W_j$ we insist $f_{j/i}$ is invertible with inverse $f_{i/j}$, or more generally

$$f_{k/j} = \frac{f_{k/i}}{f_{j/i}}$$

This effectively results from the gluing of the U_is.

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Now we want to put this into a sort of tighter package. We have like $(n+1)^2$ different functions, so there's a lot flying around. On W_i we can define a map

$$\mathcal{O}_T^{n+1} \to \mathcal{O}_T$$

where these are both somehow implicitly restricted to W_i. We know we have a distinguished sections $0, 1 \in \mathcal{O}_T$, so we can define the distinguished (global) section e_j in $\mathcal{O}_T^{n+1}$ to be 0 everywhere except the jth component. Then we send $e_j \mapsto f_{j/i}$.

Of course this is surjective since $f_{i/i} = 1$. Then the kernel of this map is as follows. Define $\mathcal{N}_i$ to be generated by $e_j = f_{j/i} e_i$. On W_j, if we just think of $\mathcal{N}_i$ as a submodule sheaf of $\mathcal{O}_T^{n+1}|_{W_i}$, if we look at another W_k, these equations $e_k = f_{k/i} e_i$ is still true, so on $W_i \cap W_j$ we will have $e_j = f_{k/j} e_j$ since $e_j = f_{j/i} e_i$. What this means is that even though $\mathcal{N}_i$ is only defined on W_i, $\mathcal{N}_i = \mathcal{N}_j$ on $W_i \cap W_j$. I.e. there is only one $\mathcal{N}$, so we can write:

$$\mathcal{L} = \mathcal{O}_T^{n+1} / \mathcal{N} \ .$$

It has generating sections $(e_0, \cdots, e_n)$ and is locally free of rank 1, i.e. isomorphic to $\mathcal{O}_T$ on the W_i.

So this means every time we have a morphism $T \to \mathbb{P}_A^n$ we get such an object $\mathcal{L}$. As it turns out, this process is also reversible. It's an algebraic fact that if we have a free module of rank 1 over a local ring with a generating collection, one of them must generate it. So this means given a locally free submodule sheaf of rank 1, the stalks are generated by single elements. But if they generate the stalks, they have to generate a neighborhood of the stalks. Now you can cover T by these open sets, and then at each of these we can write any functions in terms of the generating functions, and these will automatically satisfy the above relations, so we effectively get a map to projective space.

I.e. given a subsheaf of $\mathcal{O}_T^{n+1}$ which is locally free of rank 1, this gives a morphism $T \to \mathbb{P}^n$ such that the subsheaf we picked is the one we get with the construction we did first.

1.2. **Affine case.** This might all seem a bit abstract, so let's try to make this a bit more down to earth by letting $T = \operatorname{Spec} B$. Now we need another theorem we haven't proved:

Theorem 2. *On* $\operatorname{Spec} B$*, every quasi-coherent* $\mathcal{O}$*-module sheaf is of the form* $\tilde{M}$ *for some* B*-module* M*.*

This implies that this $\mathcal{L} = \tilde{L}$ for $L = B^{n+1}/N$ and L locally free of rank 1.

But we can actually narrow this down even more if we insist that B is a ring such that every locally free module of rank 1 is actually just free, i.e. isomorphic to B. There are plenty of examples of this. If $B = k$ is any field then the locally free modules of rank 1 are just vector spaces so they're all just k. For $B = \mathbb{Z}$ these are abelian groups, and if they're 1 they're just $\mathbb{Z}$. In general we have the following nice algebraic fact:

Fact 1. *If* B *is an integral domain, this is equivalent to* B *being a UFD.*

So we really want to write B as a quotient of B^{n+1}, i.e. we want a surjection $B^{n+1} \twoheadrightarrow B$. The ambiguity of any such map is $B^\times$, i.e we can take such a map and multiply it by any element of $B^\times$ to get another surjection. We can write these maps as (column) vectors $(b_0, \cdots, b_n)$ such that the b_i generate $(1) \subseteq B$, and this is all only well defined up to multiplication by $c \in B^\times$.

In particular, if B is a field, saying they generate the unit ideal is just saying they're not all zero, and every nonzero element is a unit, so this becomes classical projective space. So we see very explicitly morphisms $\operatorname{Spec} k \to \mathbb{P}^n_{\mathbb{Z}}$ correspond to points of projective space.

A $\mathbb{Z}$ point of projective space or a morphism $\operatorname{Spec} \mathbb{Z} \to \mathbb{P}_Z Z^n$ will be a tuple of integers which have no overall common factors, and the units are just up to signs. We can also precompose the map $\operatorname{Spec} \mathbb{Z}/p \to \operatorname{Spec} \mathbb{Z}$ or $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$.

1.3. **Affine space and hints towards** Proj**.** Classically we have a natural map $k^{n+1} \backslash 0 \to \mathbb{P}^n(k)$, but this should come from a morphism of schemes $\mathbb{A}^{n+1} \backslash 0 \to \mathbb{P}^n_A$. Recall

$$\mathbb{A}^{n+1}_A = \operatorname{Spec} A[t_0, \cdots, t_n]$$

and by zero we really mean $\operatorname{Spec} A$, so we should probably write this as:

$$\mathbb{A}^{n+1} \setminus V(t) \to \mathbb{P}^n_A \ .$$

Then we have the following open subsets:

$$\mathbb{A}^{n+1}_A \supseteq W_i = \operatorname{Spec} A\left[t_0, \cdots, t_n, t_i^{-1}\right]$$

and we want to define a map $W_i \to U_i$, where

$$U_i = \operatorname{Spec} A\left[x_{0/i}, \cdots, x_{n/i}\right] / \left(x_{i/i} - 1\right) \ .$$

To get a map $W_i \to U_i$ we need a ring homomorphism in the other direction, so we just send $x_{j/i} \mapsto t_j/t_i$. Then looking at maps $\operatorname{Spec} K \to W_i$ turns this into $k^{n+1} \setminus 0 \to \mathbb{P}^n$.

We can also think of this in the following way. The polynomial ring $A[t_0, \cdots, t_n]$ can be thought of a graded ring where the portion of a certain degree is given by the polynomials which are homogeneous of that degree. If we invert t_i this is still graded only now things can have negative grading. If an element has degree 0, this just means it can be written in terms of $t_{j/i}$s, so

$$A\left[t_0, \cdots, t_n, t_i^{-1}\right]_0 = A[t_j/t_i] \simeq A\left[x_{0/i}, \cdots, x_{n/i}\right] / \left(x_{i/i} - 1\right)$$

Remark 1. This is a hint towards the so-called Proj construction.

Now the object $\mathbb{G}_m = \operatorname{Spec} A\left[c, c^{-1}\right]$ has the structure of a group scheme, and this acts on $A\left[t_0, \cdots, t_n, t_i^{-1}\right]_0$ which we should think of as the ring of invariant functions in the sense that they're somehow constant on orbits. We will talk more about this later.