LECTURE 26

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One might be wondering if there are actually any theorems in the subject rather than just definitions and examples. We will now meet our first theorems. Actually that's not quite true, we did see that If $X = \operatorname{Spec} R$ then $\mathcal{O}_X(X) = R$ and the analogous result for modules.

1. Morphisms of affine schemes

Morphisms of affine schemes all come from Ring homomorphisms, but actually the more general statement somehow explains why this definition of Spec is so mysterious in the sense that it lets us know that the choices we made in the definition weren't arbitrary.

Theorem 1. *Let $X = \operatorname{Spec} R$ for some ring R. Let $T \in \mathbf{LRSp}$. Then every ring homomorphism $R \to \mathcal{O}_T(T)$ is actually $\varphi^\flat_X$ for a unique morphism of locally ringed spaces $\left(\varphi, \varphi^\flat, \varphi^\#\right) : T \to X$ in $\mathbf{LRSp}$.*

Before we talk about the proof we will talk a bit about what this is telling us.

Remark 1. There is a functor from ringed spaces to rings, where we send any ringed space to its global sections. This is contravariant, so it's covariant when regarded as a functor landing in $\mathbf{Ring}^{\mathrm{op}}$. Denote this by $\Gamma(-, \mathcal{O})$ where $T \mapsto \Gamma(T, \mathcal{O}_T) := \mathcal{O}_T(T)$. Then in this language the theorem says that Spec (which is a covariant functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRSp}$) is actually adjoint to $\Gamma(-, \mathcal{O})$ i.e.

$$\operatorname{Hom}_{\mathbf{LRSp}}(T, \operatorname{Spec} R) = \operatorname{Hom}_{\mathbf{Ring}^{\mathrm{op}}}(\mathcal{O}_T(T), R)$$

so Spec is right-adjoint to $\Gamma(-, \mathcal{O})$. Now since adjoint functors are unique up to unique functorial isomorphism, we actually were forced to define Spec in the way that we did.

Exercise 1. Show that the global sections functor defined on all ringed spaces has the trivial functor $R \mapsto (\cdot, R)$ as its right-adjoint.

The point is generic ringed spaces don't know about geometry. There's kind of nothing to attach the sheaf of rings to the underlying topology of the space.

Remark 2. The fact that $\mathcal{O}_X(X) = R$ means that morphisms $\operatorname{Spec} S \to \operatorname{Spec} R$ are in one-to-one correspondence with ring homomorphisms $R \to S$ so $\mathbf{AffSch} \cong \mathbf{Ring}^{\mathrm{op}}$.

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Remark 3. It is often useful to consider the category $\mathbf{Sch}/A = \mathbf{Sch}/\operatorname{Spec} A$ which has objects schemes T equipped with a morphism $T \to \operatorname{Spec} A$, and an arrow is a map $T \to T'$ such that the triangle commutes

$$\begin{array}{ccc} T & \longrightarrow & T' \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

this category is really the same as schemes in which $\mathcal{O}_T$ is a sheaf of A-algebras. But this is equivalent to making the global sections an A-algebra because we have the following diagram:

$$\begin{array}{ccc} & & \mathcal{O}_T(T) \\ & \nearrow & \downarrow \\ A & \longrightarrow & \mathcal{O}_T(U) \end{array}$$

This is important because we will interpret classical algebraic geometry as looking at $\mathbf{Sch}/k$ for $k = \overline{k}$, so the sheaf of functions is actually just a sheaf of k-algebras.

Remark 4. We can also interpret the theorem as saying what the functor represented by a locally ringed space is. The functor $\operatorname{Hom}_{\mathbf{Sch}}(-, X)$ which sends $T \to \operatorname{Hom}_{\mathbf{Sch}}(T, C)$ is the functor represented by X. If $X = \operatorname{Spec} R$, then we can say this explicitly. In particular for any ring R we can consider it as the ring $\mathbb{Z}\left[\underline{x_\alpha}\right]/(f_\beta)$. Then sending $R \to \mathcal{O}_T(T)$ means specifying where the generators go, $x_\alpha \mapsto t_\alpha$ which are specified such that $f_\beta \mapsto 0$, i.e. the t_α have to satisfy $f_\beta\left(\underline{t_\alpha}\right) = 0$.

Remark 5. For every unital cring R we can look at the subring generated by (1). This determines the unique morphism $\mathbb{Z} \to R$. This means $\mathbb{Z}$ is the initial object in the category of commutative rings with unit. Then in **LRSp** we get a unique map $T \to \operatorname{Spec} \mathbb{Z}$. In particular, $\mathbf{Sch}/\mathbb{Z} = \mathbf{Sch}$.

Remark 6. Consider a scheme, then by definition this is covered by affines. But we need to say how they are glued together, and to say that, we need to talk about the intersections. If we're lucky the intersection will be affine, but even if not this can be covered by affines. Then we can determine the morphisms between the overlaps by this affine covering. The point is that determining a scheme comes down to looking at affine schemes and their morphisms. Even morphisms of schemes can be expressed in terms of morphisms of affine schemes. Since this theorem says these are just determined from ring homomorphisms, in some sense all the data of schemes is given by morphisms of rings.

Sketch of proof. Consider $R \to \mathcal{O}_T(T)$ and now we need to construct φ. First we need the map of sets $\varphi : T \to X \operatorname{Spec} R$. From a point $p \in T$ we can take the germs $g_p : \mathcal{O}_T(T) \to \mathcal{O}_{T,p}$. Then we can precompose with $\alpha : R \to \mathcal{O}_T(T)$. Now $Q = (g_p \circ \alpha)^{-1}(m_p)$ is in $\operatorname{Spec} R$, so we define $\varphi(p) = Q$.

Now we need to show this is continuous. We will show the preimage of a closed set is closed. We know $p \in \varphi^{-1}(V(I))$ iff $I \subseteq Q$ iff $(g_p \circ \alpha)(I) \subseteq m_p$, which means this is the set of points where the maximal ideal at that point contains the stalk of $\alpha(I)$ at p: $\alpha(I)_p \subseteq m_p$. I.e. $\varphi^{-1}(V(I)) = V(\alpha(I))$ is closed.

Now we want to specify $\varphi^\flat$. Recall this automatically determines $\varphi^\#$. This is a sheaf homomorphism $\varphi^\flat : \mathcal{O}_X \to \varphi_* \mathcal{O}_T$. The idea is to give it on a basis of open sets $X_f = \operatorname{Spec} R_f$, and show it is compatible. Of course $X_f = X \setminus V(f)$, and we already saw that $\varphi^{-1}(V(f)) = V_T(\alpha(f))$

$$\varphi^{-1}(X_f) = \left\{ p \in T \mid \alpha(f)_p \in \mathcal{O}_{T,p}^X \right\}$$

if we restrict to $X_f = U$, then at the stalks $\alpha(f)$ has inverses, but then these are actually inverses in a neighborhood of that point, so this is covered by open sets where this has multiplicative inverse, but of course this is unique so this is just saying $\alpha(f)$ has an inverse in $\mathcal{O}_T(U)$. But then we have the diagram:

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \rho_{TU} \\ R_f & \dashrightarrow & \mathcal{O}_T(U) \end{array}$$

Now

$$\mathcal{O}_T(U) = (\varphi_* \mathcal{O}_T)(X_f)$$

so we are done. Now things have to be checked on X_{fg}, but this follows from them all being induced by this same α and that the following diagram commutes:

$$\begin{array}{ccccc} & & R & & \\ & \swarrow & & \searrow & \\ R_f & & & & R_g \\ & \searrow & & \swarrow & \\ & & R_{fg} & & \end{array}$$

We can also notice that $\varphi^\#$ is a map $R_Q \to \mathcal{O}_{T,p}$ defined by $\alpha \circ g_p$.

But is this unique? Suppose we have another one $(\psi, \psi^\flat, \psi^\#)$. $\psi_X^\flat$ has to be α, but we will also have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_T(T) & \xleftarrow[\psi_X^\flat = \alpha]{} & R \\ \downarrow & & \downarrow \\ \mathcal{O}_T(U) & \xleftarrow[\psi_U^\flat]{} & R_f \end{array}$$

where $U = X_f$, which commutes since $\varphi^\flat$ is a sheaf homomorphism. Then $\psi^{-1}(X_f) \subseteq T_{\alpha(f)} = T \setminus V(\alpha(f))$ which is saying $\psi(V(\alpha(f))) \subseteq V(f)$, which is saying that if the stalk $\alpha(f)_p \in \mathfrak{m}_p$, then $\psi(p) \in V(f)$. So the conclusion is that $\psi(p) = Q_1$ is some prime ideal of R which has to contain Q, but then we again have a germ map $R_{Q_1} \to \mathcal{O}_{T,p}$, and if Q_1 is not equal to Q, then $R_{Q_1} \to \mathcal{O}_{T,p}$ will not be a local homomorphism. $\square$