LECTURE 27
MATH 256A

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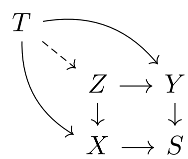
Today we will talk about total abstract nonsense.

1. Fiber products and motivation

We want to construct a fiber product $Z = X \times_S Y$ in the category of schemes. This just means we have the following diagram



Note that this is just the product in the category $\mathbf{Sch}/S$. If X, Y, and S are all affine schemes, then Z is an affine scheme, so things aren't so bad. But in general we have to do some terrible gluing process where we cover S by affines, use the preimages of this cover in X and Y as a cover of X and Y by affines, and then for every three affines construct their product and glue them together to form Z with some completely incomprehensible data. Instead we will learn a sort of gluing management theorem which comes from abstract nonsense.

Remark 1. This will actually be a useful construction throughout the course so it is worth going through it now. In particular we will use to generalize Grassmannian variety to Grassmannian schemes.

The main goal of what we are about to do is figure out when a functor is representable by a scheme. In this context the functor takes a scheme T and associates to it these schemes X and Y and the sets:

$$\underline{\underline{X}}(T) = \mathrm{Hom}_{\mathbf{Sch}}(T, X)$$

Then $T \mapsto \underline{\underline{X}}(T)$ is a functor $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$.

In this situation $\underline{\underline{X}}$ on any T is just the arrows $T \to X$ and similarly for Y, and then for every T we have that $\underline{\underline{X}}(T) \to \underline{\underline{S}}(T)$ and $\underline{\underline{Y}}(T) \to \underline{\underline{S}}(T)$ are T functorial maps. Therefore we can just define the functor represented by Z, $\underline{\underline{Z}}$ to be the fiber product as a functor $\underline{\underline{Z}} \cong \underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$

The point here is that saying Z is the fiber product in $\mathbf{Sch}$ can be rephrased as saying that the functor it represents is the fiber product in $\mathbf{Set}$ on any T.

Theorem 1. *A functor $F : \mathbf{Sch}^{op} \to \mathbf{Set}$ is representable (i.e. $F \cong \underline{\underline{X}}$) iff*

(1) F is a sheaf (in the Zariski topology on $\mathbf{Sch}$)
(2) F can be covered by representable open subfunctors

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2. Yoneda lemma

2.1. **Representable functors.** So consider a scheme Z, and associate to it the functor it represents:

$$\underline{\underline{Z}} = \mathrm{Hom}_{\mathbf{Sch}}(-, Z)$$

which is of course a functor $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$.

Example 1. Let k be a field, then $\underline{\underline{Z}}(\operatorname{Spec} k)$ sometimes written $Z(k)$ maps to the set consisting of maps $\operatorname{Spec} k \to Z$ which all must map points $\mathrm{pt} \to p \in Z$. Then we have the residue field $\mathcal{O}_{Z,p}/m_p = k_p \hookrightarrow k$ so this is somehow just points, and this construction is more geometric than we might think.

Example 2. For an affine scheme $X = \operatorname{Spec} R$, from the theorem last time we have that

$$\underline{\underline{X}}(T) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_T(T))$$

and in fact this is a functor since for any $\varphi T' \to T$ we get $\varphi^\flat : \mathcal{O}_T(T) \to \mathcal{O}_{T'}(T')$ and we can just postcompose.

2.2. **Functorial maps and the Yoneda functor.** We now specify what a functorial map is. Note that functors between two categories form a category.[1] The arrows in this category are functorial maps (or natural transformations) which, for any two functors F and G, are arrows $\varphi : F \to G$ such that for every object T in the initial category, we get a map $\varphi_T : F(T) \to G(T)$ which is functorial in T in the sense that for any $\alpha : T' \to T$ the following commutes:

$$\begin{array}{ccc} F(T) & \xrightarrow{\varphi_T} & G(T) \\ {\scriptstyle F(\alpha)}\uparrow & & \uparrow{\scriptstyle G(\alpha)} \\ F(T') & \xrightarrow{\varphi_{T'}} & G(T') \end{array}$$

Warning 1. In our particular case we're coming from $\mathbf{Sch}^{\mathrm{op}}$ so α would have actually been in the other direction.

So when we send objects of a category to the functor it represents, this will turn out to be a functor. In our context, when we send $X \mapsto \underline{\underline{X}}$, we are sending an object in $\mathbf{Sch}$ to an object in $\mathbf{Functors}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set})$. The map sending $X \mapsto \underline{\underline{X}}$ is a functor since for any map $X \to Y$ we get a map $\underline{\underline{X}}(T) \to \underline{\underline{Y}}(T)$ given by:

$$(T \to X) \mapsto \left(\begin{array}{ccc} T & \longrightarrow & X \\ & \searrow & \downarrow \\ & & Y \end{array} \right)$$

This is called the Yoneda functor.

2.3. **Yoneda lemma.** The Yoneda functor comes with the following:

Lemma 1 (Yoneda (weak))**.** *The Yoneda functor $X \mapsto \underline{\underline{X}}$ is an equivalence of categories from any category $\mathbf{S}$ to its image in $\mathbf{Functors}(\mathbf{S}^{op}, \mathbf{Set})$. This induces a bijection on arrows between $X \to Y$ and $\underline{\underline{X}} \to \underline{\underline{Y}}$, so every functorial map between two representable functors comes from a unique map between the two objects which gave rise to them.*

[1] Roughly speaking the category of categories are 2-categories since the functors form a category as well.

We strengthen this a bit before proving it:

Lemma 2 (Yoneda)**.** *For an arbitrary functor* $F : \mathbf{S}^{op} \to \mathbf{Set}$ *the maps* $\underline{\underline{X}} \to F$ *are in canonical correspondence with* $F(X)$*.*

This can be seen as somehow saying that the functor represented by the functor F in **Functors** is somehow F itself.

Proof. For $f \in F(X)$ we want a functor map $\underline{\underline{X}} \to F$, so for every T we need a map $\underline{\underline{X}}(T) \to F(T)$. To do this we take a map $\varphi : T \to X$, then we take $F(\varphi)$ and evaluate at f to get $F(\varphi)(f) \in F(T)$.

In the other direction, given a functorial map $\underline{\underline{X}} \to F$ we evaluate at X to get $\underline{\underline{X}}(X) \to F(X)$. Since $\underline{\underline{X}}(X)$ has the distinguished element id_X, the image of id_X under this map gives us an element of $F(X)$, and the rest of the proof is just diagram chasing. □

3. Back to sheaves and schemes

There's a sense in which any functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is a sort of presheaf on **Sch**. To see this we can restrict this to the open subschemes $U \subseteq T$, where the arrows are inclusions, and then this is a presheaf of sets on T. Explicitly this sends $U \to F(U)$, and the inclusion maps $U \subseteq V$ are reversed to give the restriction maps on this presheaf. Then we say F is a sheaf if this presheaf is a sheaf for all T.

Lemma 3. *The functor represented by a scheme* $\underline{\underline{X}}$ *is always a sheaf.*

Proof. This says that given a scheme T, an open covering

$$T = \bigcup_{\alpha} U_\alpha \ ,$$

and maps $f_\alpha : U_\alpha \to X$ (i.e. $f_\alpha \in \underline{\underline{X}}(U_\alpha)$) which are compatible in the sense that f_α and f_β agree on $U_\alpha \cap U_\beta$, then we have a unique map $f : T \to X$ such that $f_\alpha = f|_{U_\alpha}$. □