LECTURE 28
MATH 256A

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Today will be abstract nonsense part two. Recall we have the following theorem:

Theorem 1. *A functor $F : \mathbf{Sch}^{op} \to \mathbf{Set}$ is representable (i.e. $F \cong \underline{\underline{X}}$) iff*

(1) *F is a sheaf*
(2) *F can be covered by representable open subfunctors*

We can easily see these conditions are necessary, so the content is really showing sufficiency.

1. Subfunctors

1.1. **Definitions.** Given $W \subset Z$ an open subset[1] the question is how $\underline{\underline{W}}$ and $\underline{\underline{Z}}$ are related. One way is that since $W \hookrightarrow Z$ is an inclusion of schemes, there is a corresponding functorial map $\underline{\underline{W}} \to \underline{\underline{Z}}$ which turns out to be injective as well. This just means $\underline{\underline{W}}(T) \hookrightarrow \underline{\underline{Z}}(T)$ for any T. So given $T \to W$, we can compose with $W \hookrightarrow Z$, and this determines the map $T \to Z$ uniquely as in the following diagram:

$$\begin{array}{ccc} T & \longrightarrow & W \\ & \searrow & \downarrow \\ & & Z \end{array}$$

We can also ask the inverse question: if we are given a map $T \to Z$, then the corresponding map $T \to W$ that makes the diagram commute:

$$\begin{array}{ccc} & & W \\ & \nearrow & \downarrow \\ T & \xrightarrow{\varphi} & Z \end{array}$$

might not exist, but if it does it is unique. In particular, this exists iff $\varphi(T) \subseteq W$. We say that $\underline{\underline{W}}$ is a subfunctor of $\underline{\underline{Z}}$. More explicitly,

$$\underline{\underline{W}}(T) = \left\{ (\varphi : T \to Z) \in \underline{\underline{Z}}(T) \mid \varphi(T) \subseteq W \right\}$$

But now if we take some T' which doesn't satisfy this, then we have the following diagram

$$\begin{array}{ccc} \varphi^{-1}(W) = U & \longrightarrow & W \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & Z \end{array}$$

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[1] Or in general, a monomorphism.

which is a fiber square exactly when $\varphi \circ \alpha(T) \subseteq W$ because

$$\begin{array}{ccccc} & & W & & \\ & \overset{!}{\nearrow} & & \searrow & \\ T' & & & & Z \\ & \underset{\alpha}{\searrow} & & \underset{\varphi}{\nearrow} & \\ & & T & & \end{array}$$

But this is equivalent to

$$\alpha(T') \subseteq \varphi^{-1}(W) = U$$

so we have the following:

$$\underline{\underline{U}}(T') = \underline{\underline{T}}(T') \times_{\underline{\underline{Z}}(T')} \underline{\underline{W}}(T')$$

Definition 1. A subfunctor $G \subseteq F$ is open if for every $T \in \mathbf{Sch}$, and $\underline{\underline{T}} \to F$ (equivalently $f \in F(T)$) we have that $\underline{\underline{T}} \times_F G = \underline{\underline{U}}$ for an open $U \subseteq T$.

Given $G \subseteq F$ open, and $\underline{\underline{T}} \to F$, we now get an open subscheme $U \subseteq T$. But what would need to be the case for $U = T$? As it turns out, this is the case iff $f \in G(T)$. The functorial reason is somehow the same as the geometric one. If $f \in G(T)$, then the morphism factors as

$$\begin{array}{ccc} & \nearrow & G \\ & & \big\downarrow \\ \underline{\underline{T}} & \longrightarrow & F \end{array}$$

so $\underline{\underline{U}} = \underline{\underline{T}}$ and therefore $U = T$ by Yoneda.

1.2. **Lemmas.**

Lemma 1. *A map of representable functors $\underline{\underline{W}} \to \underline{\underline{Z}}$ is an open embedding iff $\underline{\underline{W}}$ is an open subfunctor of $\underline{\underline{Z}}$.*

Lemma 2. *If $G, H \subset F$ are subfunctors, and G is open then $G \cap H \subseteq H$ is also open.*

Proof. In this situation we have a functorial diagram:

$$\begin{array}{ccc} G \cap H & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ H & \longrightarrow & F \end{array}$$

and then we take $\underline{\underline{U}}$ such that the larger rectangle is a fiber square:

$$\begin{array}{ccccc} \underline{\underline{U}} & \longrightarrow & G \cap H & \longrightarrow & G \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \underline{\underline{T}} & \longrightarrow & H & \longrightarrow & F \end{array}$$

Then the question is whether or not the smaller square on the left is a fiber square, and since this is true for sets it is true here. $\square$

1.3. **Open subfunctors and covers.**

Definition 2. Open subfunctors $G_\alpha \subseteq F$ cover F if for every functorial map $\underline{\underline{T}} \to F$ (equivalently $f \in F(T)$) the open sets $U_\alpha \subseteq T$ such that $\underline{\underline{U_\alpha}} = G \times_F \underline{\underline{T}}$ cover T.

Warning 1. In general

$$F(T) \neq \bigcup_\alpha G_\alpha(T)$$

This is true sometimes though, as in the following example.

Example 1. If $T = \operatorname{Spec} k$, then

$$F(\operatorname{Spec} k) = \bigcup_\alpha G_\alpha(\operatorname{Spec} k)$$

For arbitrary $\underline{\underline{T}} \to F$ we get these U_α and we want to see if they cover T. For any point $p \in T$ we can take the residue field $\mathcal{O}_{T,o}/\mathfrak{m}_p = k_p$ and then we have a morphism $\operatorname{Spec} k_p \to T$ which sends pt $\mapsto p$. Therefore $\underline{\underline{T}}(\operatorname{Spec} k)$ can be thought of as pairs

$$\underline{\underline{T}}(\operatorname{Spec} k) = \{(p, k_p \hookrightarrow k)\}$$

So take any $p \in T$, represent it as a map $\operatorname{Spec} k \to \underline{\underline{T}}$, then we can just compose this to get $\operatorname{Spec} k \to F$, so it must factor through U_α. Therefore we have

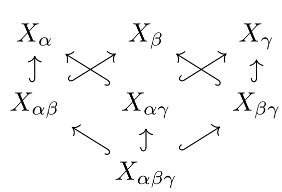
$$F(\operatorname{Spec} k) = \bigcup_\alpha G_\alpha(\operatorname{Spec} k)$$

for all k iff F being covered by the G_αs.

2. Proof of the theorem

We now use our results about subfunctors to prove the theorem:

Proof of the theorem. Let $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be covered by the G_α where $G_\alpha \cong \underline{\underline{X_\alpha}}$. By the lemmas, $G_\alpha \cap G_\beta \subseteq G_\alpha$ is open, and since this is representable as well, $G_\alpha \cap G_\beta = \underline{\underline{X_{\alpha,\beta}}}$ for some $X_{\alpha,\beta} \subset X_\alpha$. Then the diagram is:



and the cocycle condition is that we can always put an $X_{\alpha,\beta,\gamma}$ such that every diamond is a fiber square. So now we get X by gluing, but we aren't sure that this is actually F.

We know all of the $\underline{\underline{X_\alpha}} = G_\alpha \hookrightarrow F$, and then Yoneda gives us an identification $\underline{\underline{X_\alpha}} \to F$ with some $f_\alpha \in F(X_\alpha)$ which are compatible on $X_\alpha \cap X_\beta$. Now the fact that F is a sheaf means there is some $f \in F(X)$ which gives us $\underline{\underline{X}} \to F$ which restricts to all of the X_α to give $f_\alpha : \underline{\underline{X_\alpha}} \to F$. For $g \in F(T)$ this corresponds to $\underline{\underline{T}} \to F$ and since the $U_\alpha \subset T$ cover, we have

$$\begin{array}{ccc} \underline{\underline{U}}_\alpha & \longrightarrow & F \\ & \searrow & \uparrow \\ & & \underline{\underline{X_\alpha}} \longrightarrow X \end{array}$$

so we get a map $T \to X$. Then we have to check these are actually the inverses of one another. $\square$

We will use this theorem a few times, it's a handy thing to have around. One thing we will use this for is to show fiber products of schemes exist. That is, we will show that given the diagram:

$$\begin{array}{ccc} X & & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

there is some $\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ which is representable. Instead of doing some messy gluing, we can use this theorem.