LECTURE 29

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Nikolay's notes

1. Fiber product

Let X and Y be schemes over S. Then we want to consider the fiber product $\underline{\underline{Z}} \cong \underline{\underline{C}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ which fits into the square

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X & & & & Y \\ & \searrow^{p} & & \swarrow_{q} & \\ & & S & & \end{array}$$

To see this is representable, we will apply the theorem from last time. Recall this said that in order for a functor $\underline{\underline{Z}}$ to be representable, we need it to be a sheaf, and we need it to be covered by representable open subfunctors.

Proof. To see if $\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ is a sheaf, note we have

$$U \to X(U) \times_{S(U)} Y(U) \subseteq X(U) \times Y(U)$$

and the equalizer condition $X, Y \to S$ is a local condition since S is a sheaf.

To check that this is covered by representable open subfunctors, notice that we have the two open embeddings:

$$\begin{array}{ccccc} V \subseteq p^{-1}(U) & & & & W \subseteq q^{-1}(U) \\ & \searrow & & \swarrow & \\ & & U & & \end{array}$$

for $U \subseteq S$ open.

Claim 1. $\underline{\underline{W}} \times_{\underline{\underline{U}}} \underline{\underline{W}}$ is an open subfunctor of $\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$

Given $\underline{\underline{T}} \to \underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ we want something such that

$$\underline{\underline{T}} \leftarrow ? \to \underline{\underline{V}} \times_{\underline{\underline{U}}} \underline{\underline{W}}$$

We have that $\varphi \in \underline{\underline{T}}(T')$ is in ? iff

$$\varphi(T') \subseteq \alpha^{-1}(V) \cap \beta^{-1}(W)$$

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Such $\underline{\underline{V}} \times_{\underline{\underline{U}}} \underline{\underline{W}}$ for U, V, W affine cover $\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ Since for $\left(\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}\right)(\operatorname{Spec} k)$,

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & & \searrow & \\ \operatorname{Spec} k & & & & S \\ & \searrow & & \nearrow & \\ & & Y & & \end{array}$$

Now we're done, because we showed we can cover $\underline{\underline{X}} \times_{\underline{\underline{S}}} \underline{\underline{Y}}$ on the level of sets. □

Now suppose A and B are R-algebras.

Claim 2. $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$.

Proof. If we have maps

$$T \to \operatorname{Spec}(A \otimes_R B)$$

then this means

$$A \otimes_R B \to \mathcal{O}_T(T)$$

Also, if we have

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & & \searrow & \\ T & & & & S \\ & \searrow & & \nearrow & \\ & & Y & & \end{array}$$

this implies there exists a map $R \to \mathcal{O}_T(T)$, so we have R-algebra homomorphisms $A \to \mathcal{O}_T T$ and $B \to \mathcal{O}_T(T)$.

Now we want to show that these things are equivalent. This follows from the universal property. Explicitly we have a coproduct of R-algebras

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow^{\alpha} & \\ A \otimes_R B & \overset{\exists!}{\dashrightarrow} & C \\ \uparrow & \nearrow_{\beta} & \\ B & & \end{array}$$

Note that $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ is well-defined since the RHS is multi-linear over R. Then we can map:

$$\begin{array}{ccc} A \longrightarrow A \otimes_R B & & B \longrightarrow A \otimes_R B \\ a \longmapsto a \otimes 1 & & b \longmapsto 1 \otimes b \end{array}$$

so we can send

$$\begin{array}{c} A \otimes_R B \longrightarrow C \\ \alpha \otimes \beta \longmapsto \alpha(a)\beta(b) \end{array}$$

□

Exercise 1. Show that $X \times_S Y$ is the product in **Sch**$/S$.

Example 1. For $S = \operatorname{Spec} k$, X and Y are varieties over k.

Let $R = \mathcal{O}(X)$ and $S = \mathcal{O}(Y)$, $X^S := \operatorname{Spec} R$ and $Y^S := \operatorname{Spec} S$. Note that:

$$\left(X^S\right)_{\mathrm{cl}} = C = \left\{ \begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} R \\ & \searrow \quad \nearrow & \\ & \operatorname{Spec} k & \end{array} \right\}$$

In this case we have the following:

$$\left(X^S \times_{\operatorname{Spec} k} Y^S\right)_{\mathrm{cl}} = X \times Y$$

This is obvious, there's no need for the nullstellensatz. What is less obvious is the fact that:

$$\operatorname{Spec}\left(\mathcal{O}\left(X \times Y\right)\right) = X^S \times Y^S$$

So, is

$$\mathcal{O}\left(X + Y\right) \cong \mathcal{O}\left(X\right) \otimes_R \mathcal{O}\left(Y\right) \ ?$$

Take $X \subseteq k^m$ and $Y \subseteq k^n$. Then

$$\mathcal{O}\left(k^m\right) = k\left[x_1, \cdots, x_m\right] \qquad \mathcal{O}\left(k^n\right) = k\left[y_1, \cdots, y_n\right]$$
$$\mathcal{I}\left(X\right) = \left(f_1, \cdots, f_k\right) \qquad \mathcal{I}\left(Y\right) = \left(g_1, \cdots, g_l\right)$$

and therefore

$$\mathcal{O}\left(X\right) \otimes_R \mathcal{O}\left(Y\right) = k\left[x_1, \cdots, x_m, y_1, \cdots, y_n\right] / J$$

where we have set:

$$J = \left(f_1\left(\underline{x}\right) \cdots f_k\left(\underline{x}\right), g\left(\underline{y}\right) \cdots g_l\left(\underline{Y}\right)\right)$$

which implies $V\left(J\right) = X \times Y = k^m \times k^n$.

In general,

$$X^S \times Y^S = \operatorname{Spec}\left(\mathcal{O}\left(X\right) \otimes_R \mathcal{O}\left(Y\right)\right)$$

contains

$$\left(X^S \times_{\operatorname{Spec}} Y^S\right)_{\mathrm{red}} = \operatorname{Spec}\left(\mathcal{O}\left(X\right) \otimes_R \mathcal{O}\left(Y\right) / \sqrt{0}\right)$$

If A and B are reduced k-algebras (k dg closed), then $A \otimes_k B$ is reduced.

Example 2. We offer an example of why this is nontrivial. Let $\Gamma k = p$, and $\alpha \in k$ such that $\alpha \notin k^p$, i.e. α is a pth root. Then

$$k = k\left[\alpha^{1/p}\right] = k\left[x\right] / \left(x^p - \alpha\right)$$

and then

$$k \otimes_k k = k\left[x\right] / \left(x^p - \alpha\right) = k\left[x\right] / \left(x - \alpha^{1/p}\right)^p$$

$x - \alpha^{1/p}$ is nilpotent, so k must be a perfect field, i.e. it must have a pth root.