

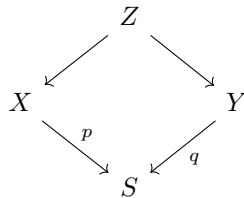
LECTURE 29

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Nikolay's notes

1. FIBER PRODUCT

Let X and Y be schemes over S . Then we want to consider the fiber product $\underline{Z} \cong \underline{X} \times_{\underline{S}} \underline{Y}$ which fits into the square



To see this is representable, we will apply the theorem from last time. Recall this said that in order for a functor \underline{Z} to be representable, we need it to be a sheaf, and we need it to be covered by representable open subfunctors.

Proof. To see if $\underline{X} \times_{\underline{S}} \underline{Y}$ is a sheaf, note we have

$$U \rightarrow X(U) \times_{S(U)} Y(U) \subseteq X(U) \times Y(U)$$

and the equalizer condition $X, Y \rightarrow S$ is a local condition since S is a sheaf.

To check that this is covered by representable open subfunctors, notice that we have the two open embeddings:

$$\begin{array}{ccc}
 V \subseteq p^{-1}(U) & & W \subseteq q^{-1}(U) \\
 & \searrow & \swarrow \\
 & U &
 \end{array}$$

for $U \subseteq S$ open.

Claim 1. $\underline{W} \times_{\underline{U}} \underline{W}$ is an open subfunctor of $\underline{X} \times_{\underline{S}} \underline{Y}$

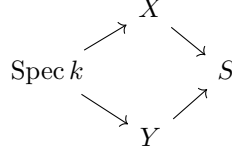
Given $\underline{T} \rightarrow \underline{X} \times_{\underline{S}} \underline{Y}$ we want something such that

$$\underline{T} \xleftarrow{\alpha} \underline{V} \times_{\underline{U}} \underline{W}$$

We have that $\varphi \in \underline{T}(T')$ is in ? iff

$$\varphi(T') \subseteq \alpha^{-1}(V) \cap \beta^{-1}(W)$$

Such $\underline{V} \times_U \underline{W}$ for U, V, W affine cover $\underline{X} \times_S \underline{Y}$ Since for $(\underline{X} \times_S \underline{Y}) (\text{Spec } k)$,



Now we're done, because we showed we can cover $\underline{X} \times_S \underline{Y}$ on the level of sets. \square

Now suppose A and B are R -algebras.

Claim 2. $X \times_S Y = \text{Spec}(A \otimes_R B)$.

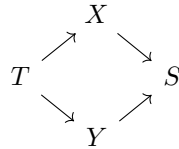
Proof. If we have maps

$$T \rightarrow \text{Spec}(A \otimes_R B)$$

then this means

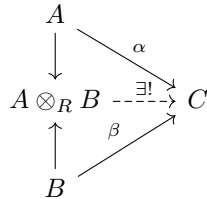
$$A \otimes_R B \rightarrow \mathcal{O}_T(T)$$

Also, if we have



this implies there exists a map $R \rightarrow \mathcal{O}_T(T)$, so we have R -algebra homomorphisms $A \rightarrow \mathcal{O}_T(T)$ and $B \rightarrow \mathcal{O}_T(T)$.

Now we want to show that these things are equivalent. This follows from the universal property. Explicitly we have a coproduct of R -algebras



Note that $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ is well-defined since the RHS is multi-linear over R . Then we can map:

$$A \rightarrow A \otimes_R B \quad B \rightarrow A \otimes_R B$$

$$a \rightarrow a \otimes 1 \quad b \rightarrow 1 \otimes b$$

so we can send

$$A \otimes_R B \longrightarrow C$$

$$\alpha \otimes \beta \longrightarrow \alpha(a) \beta(b)$$

\square

Exercise 1. Show that $X \times_S Y$ is the product in **Sch**/ S .

Example 1. For $S = \text{Spec } k$, X and Y are varieties over k .

Let $R = \mathcal{O}(X)$ and $S = \mathcal{O}(Y)$, $X^S := \text{Spec } R$ and $Y^S := \text{Spec } S$. Note that:

$$(X^S)_{\text{cl}} = C = \left\{ \begin{array}{ccc} \text{Spec } k & \longrightarrow & \text{Spec } R \\ & \searrow & \nearrow \\ & \text{Spec } k & \end{array} \right\}$$

In this case we have the following:

$$(X^S \times_{\text{Spec } k} Y^S)_{\text{cl}} = X \times Y$$

This is obvious, there's no need for the nullstellensatz. What is less obvious is the fact that:

$$\text{Spec}(\mathcal{O}(X \times Y)) = X^S \times Y^S$$

So, is

$$\mathcal{O}(X + Y) \cong \mathcal{O}(X) \otimes_R \mathcal{O}(Y) ?$$

Take $X \subseteq k^m$ and $Y \subseteq k^n$. Then

$$\begin{aligned} \mathcal{O}(k^m) &= k[x_1, \dots, x_m] & \mathcal{O}(k^n) &= k[y_1, \dots, y_n] \\ \mathcal{I}(X) &= (f_1, \dots, f_k) & \mathcal{I}(Y) &= (g_1, \dots, g_l) \end{aligned}$$

and therefore

$$\mathcal{O}(X) \otimes_R \mathcal{O}(Y) = k[x_1, \dots, x_m, y_1, \dots, y_n] / J$$

where we have set:

$$J = (f_1(\underline{x}) \cdots f_k(\underline{x}), g(\underline{y}) \cdots g_l(\underline{y}))$$

which implies $V(J) = X \times Y = k^m \times k^n$.

In general,

$$X^S \times Y^S = \text{Spec}(\mathcal{O}(X) \otimes_R \mathcal{O}(Y))$$

contains

$$(X^S \times_{\text{Spec } k} Y^S)_{\text{red}} = \text{Spec}(\mathcal{O}(X) \otimes_R \mathcal{O}(Y) / \sqrt{0})$$

If A and B are reduced k -algebras (k dg closed), then $A \otimes_k B$ is reduced.

Example 2. We offer an example of why this is nontrivial. Let $\Gamma k = p$, and $\alpha \in k$ such that $\alpha \notin k^p$, i.e. α is a p th root. Then

$$k = k[\alpha^{1/p}] = k[x] / (x^p - \alpha)$$

and then

$$k \otimes_k k = k[x] / (x^p - \alpha) = k[x] / (x - \alpha^{1/p})^p$$

$x - \alpha^{1/p}$ is nilpotent, so k must be a perfect field, i.e. it must have a p th root.