LECTURE 30
MATH 256A

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1. Examples and applications of products

1.1. **Products of classical varieties.** One of the things that we haven't seen yet, but we will, is that classical algebraic varieties over an algebraically closed field k are equivalent to reduced (no non-zero nilpotent elements) schemes locally of finite type over the same k. In particular, given a scheme X, this will correspond to the classical variety:

$$X_{\mathrm{cl}} = X(k) = \underline{\underline{X}}_k (\operatorname{Spec} k)$$

which should be thought of as the k-points of X. The significance of

$$X \times_k Y \coloneqq X \times_{\operatorname{Spec} k} Y$$

is that it is the product in the category $\mathbf{Sch}/k$, and in particular its k-points, will just be the product of morphisms to X with morphisms of Y:

$$(X \times_k Y)_{\mathrm{cl}} = X(k) \times Y(k)$$

This is effectively built into the definition.

Let $X = \operatorname{Spec} R$ where $R = k[x_1, \cdots, x_n] / \mathcal{I}(x)$, and $Y = \operatorname{Spec} S$ where $S = k[y_1, \cdots, y_m] / \mathcal{I}(Y)$. Then

$$X \times_k Y = \operatorname{Spec} R \otimes_k S$$

where

$$R \otimes_k S = k[x_1, \cdots, x_n, y_1, \cdots, y_m] / (\mathcal{I}(x), \mathcal{I}(y))$$

this is a good candidate for the classical variety corresponding to $X \times_k Y$. But the non-obvious part of this is that this ideal might not be the whole ideal we need. So the question is whether or not $X \times_k Y$ is reduced in this situation, i.e. are the natural equations of $X_{\mathrm{cl}} \times Y_{\mathrm{cl}}$ "all" of the equations? This question just comes down to the affine case. The following theorem addresses this:

Theorem 1. *If k is a perfect field, and A and B are reduced k-algebras, then $A \otimes_k B$ is reduced.*

Example 1. Consider $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Then we have:

$$\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \operatorname{Spec} \mathbb{C} \otimes_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$$

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We can think of $\mathbb{C} = \mathbb{R}[x] / \left(x^2 - 1\right)$ and then

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &= \mathbb{R}[x, y] / \left(x^2 + 1, y^2 + 1\right) \\ &= \mathbb{C}[y] / \left(y^2 + 1\right) \\ &= \mathbb{C}[y] / \left(\left(y^2 + i\right)\left(y^2 - i\right)\right) \\ &\cong \mathbb{C} \times \mathbb{C} \end{aligned}$$

Example 2. Let $k = \mathbb{F}_p(\alpha)$ be the field with p elements adjoined with a single variable. This is not perfect, because there is no pth root of α. Then let $K = \mathbb{F}_p\left(\alpha^{1/p}\right) = k[x] / (x^p - \alpha)$. Then

$$K \otimes_k K = k[x, y] / (x^p - \alpha, y^p - \alpha) = k[y] / (y^p - \alpha) = k[y] / \left(y - \alpha^{1/p}\right)^p$$

which is not reduced, since $y - \alpha^{1/p}$ is a nonzero nilpotent element.

If a ring is reduced, this means $\sqrt{0} = 0$. But

$$\sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$$

so if we look at

$$A \to \prod_{\mathfrak{p}} (A/\mathfrak{p})$$

then saying this is reduced is just saying this map is injective. We can further embed this into its function field to get

$$A \hookrightarrow \prod_{\mathfrak{p}} (A/\mathfrak{p}) \hookrightarrow \prod_{\mathfrak{p}} K(A/\mathfrak{p})$$

So if A, B are reduced, then they each embed in some product of fields $A \hookrightarrow \prod K_i$, and $B \hookrightarrow \prod L_i$. Therefore we have the following embedding:

$$A \otimes_k B \hookrightarrow \left(\prod K_i\right) \otimes_k \left(\prod L_i\right)$$

The tensor product doesn't commute with infinite products, but we can just use the general fact that this embeds inside:

$$\left(\prod K_i\right) \otimes_k \left(\prod L_i\right) \hookrightarrow \prod (K_i \otimes_k L_j)$$

Therefore this all comes down to the example $K \otimes_k L$ where K, L are extensions of k. Then the actual theorem is:

Proposition 1. *$K \otimes_k L$ is reduced if $K \supset k$ is a separable extension.*

1.2. **How taking fiber products affects morphisms.** There are various ways to think about fiber products depending on which morphism you think is important. One way to think of this is that f is the important morphism, and Y is just some scheme over S, and then we want to think about this as a change of base:

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & S \end{array}$$

So the map $f' : Z \to Y$ is somehow f with base extended to Y. There are many theorems in algebraic geometry which somehow say that when f is somehow nice, so is f'. We'll consider some examples of these situations.

Example 3. Consider an open embedding $U \hookrightarrow X$ and $Y \to X$. Then $f^{-1}(U)$ is just the fiber product.

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

This is also an open embedding, so the idea is that open embeddings are preserved under base extensions.

A similar example, is that if we have two open embeddings, their intersection is the fiber product:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ V & \hookrightarrow & X \end{array}$$

Example 4. We can also add a variable to a ring, $A[t_1, \cdots, t_n]$, or just $A[t]$, which makes sense in the scheme theoretic language in the sense that for any ring homomorphism $A \to B$ then $B \otimes_A A[t] = B[t]$. In this case the diagram is:

$$\begin{array}{ccc} \operatorname{Spec} B[t] & \longrightarrow & \operatorname{Spec} A[t] \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \end{array}$$

In particular, this means

$$\operatorname{Spec} B[t] = \operatorname{Spec} B \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}[t] = \operatorname{Spec} B \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$$

Recall we had this gluing construction of $\mathbb{P}^n_A$ where we glued a bunch of Specs of polynomial rings with coefficients in A, and then

$$\mathbb{P}^n_A = \operatorname{Spec} A \times \mathbb{P}^n_{\mathbb{Z}}$$

Example 5. Let's say we want to do a base extension of a closed embedding $Y \to X$. This is locally $\operatorname{Spec} R$, and then we have $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$. Now we base extend this:

$$\begin{array}{ccc} T \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

The picture of this locally is when T and X are both affine. So if T locally looks like $\operatorname{Spec} S$ and X looks like $\operatorname{Spec} R$, then the situation is that we have morphisms $\varphi : R \to S$, and

$$S \otimes_R R/I = S/(\varphi(I))$$

so the resulting morphism will also be a closed embedding.