# LECTURE 31
# MATH 256A

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## 1. Fiber products

We will explore some theorems about fiber products today. One thing that is tricky, is that it isn't generally the same as the fiber product of the underlying topological spaces.

If we have a diagram of schemes

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T \end{array}$$

where $Z = X \times_T Y$ then this is of course also a diagram of topological spaces. There is a unique map from $Z$ to the topological fiber product just by the universal property, but this is not bijective in general.

**Example 1.** In particular if we take $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$, then since $k[x] \otimes_k k[y] = k[x, y]$, we have $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ which maps to the topological fiber product.

**Example 2.** Consider $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. Topologically this is just the fiber product of two points over a point, which is of course a point. But as schemes this fiber product is $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = V\left(x^2+1\right) \subseteq \mathbb{A}^1_{\mathbb{C}}$ which has two points.

So this unique map isn't injective even if $X \to T$ and $Y \to T$ are injective.

**Theorem 1.** *The unique map from $Z \to X \times_T^{top} Y$ is always surjective.*

*Proof.* Whenever we have a point $x \in X$ of a scheme, we can consider the residue field at $x$ $\mathcal{O}_{X,X}/\mathfrak{m}_x = k(x)$. Then to give a morphism $\varphi : \operatorname{Spec} K \to X$ is exactly the same as giving a point $x \in X$ and a field extension of $k(x)$ over $K$. This is because any such $\varphi : \mathrm{pt} \mapsto x \in X$ gives us $\varphi^\# : \mathcal{O}_{X,x} \to K$, which has to send the maximal ideal in this ring to the zero ideal in $K$, so the kernel has to be the maximal ideal. I.e. it has to factor through $k(X)$ to be a morphism of locally ringed spaces:

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & & \longrightarrow & & K \\ & \searrow & & \nearrow & \\ & & k(x) & & \end{array}$$

Conversely, we have this picture and we can define $\varphi^\#$ this way and that's a morphism.

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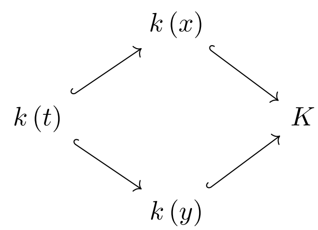
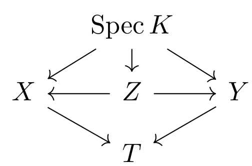
So to prove this, look at $x \in X$ and $y \in Y$ which both map to the same $t \in T$. Then given these maps $X \to T$ and $Y \to T$ we get a map $\mathcal{O}_{T,t} \to \mathcal{O}_{T,x}$ which induces an extension $k(t) \hookrightarrow k(x)$. Similarly for $Y$ we get $k(t) \hookrightarrow k(y)$. So we get




each of these extensions give us the diagram:




□

## 2. Monomorphisms

**Definition 1.** A morphism $\varphi : X \to Y$ in any category is a monomorphism iff the corresponding functorial map of the functors is injective. I.e. for all $T$, $\underline{\underline{X}}(T) \hookrightarrow \underline{\underline{Y}}(T)$.

More explicitly, two elements in $\underline{\underline{X}}(T)$ are just maps $\alpha, \beta : T \to X$, and then we compose with $\varphi$ to get maps to $Y$, so the above is equivalent to saying that $\varphi \circ \alpha = \varphi \circ \beta$ implies that $\alpha = \beta$.

**Example 3.** In **Top** and **Grp** the monomorphisms are exactly the injective maps. In **Sch** every monomorphism will be injective, however not every injective morphism of schemes will be a monomorphism.

There are three general types of examples of monomorphisms for schemes:

**Example 4.** Consider an open embedding $U \hookrightarrow X$ of schemes. We really have the identification:

$$\underline{\underline{U}}(T) = \{\varphi : T \to X \mid \varphi(T) \subseteq U\} \hookrightarrow \underline{\underline{X}}(T)$$

are monomorphism.

**Example 5.** Consider a closed embedding $i : Z \hookrightarrow X$. This means that topologically it's a closed embedding, but also that $i^{\#} : i^{-1}\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ is surjective.

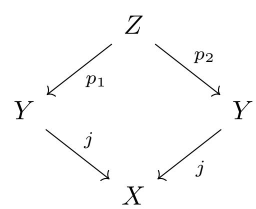
If we have two maps $\alpha, \beta : T \to Z$, then since $i$ is an injective map, if $i \circ \alpha = i \circ \beta$ then $\alpha$ and $\beta$ are the same as maps of sets. They might not be the same as maps of ringed spaces, so we still have to check this. For any $t \in T$, $\alpha(t) = \beta(t) = x \in Z \subseteq X$. Then $i^{\#}_x : \mathcal{O}_{X,x} \to \mathcal{O}_{Z,x}$ and we have $\alpha^{\#}_t$ and $\beta^{\#}_t$ both mapping $\mathcal{O}_{Z,x} \to \mathcal{O}_{T,t}$. So if $\alpha^{\#}_t \circ i^{\#} = \beta^{\#}_t \circ i^{\#}$, then since this is a closed embedding, we have $\alpha^{\#}_t = \beta^{\#}_t$ as desired.

**Example 6.** The third example is as follows. Consider $S^{-1}R$ for $S \subseteq R$ some multiplicative subset. Then we claim $\operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$ is a monomorphism. In particular, for $x \in X = \operatorname{Spec} R$ we get a map from $R \to \mathcal{O}_{X,x} = R_x$ so we get a map $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} R$ which is a monomorphism.

Let's see why this is a monomorphism. We could do this by going back to stalks, but we will use the fact that we know exactly how to characterize morphisms from any scheme to an affine scheme. We know morphism $T \to \operatorname{Spec} R$ are the same as morphisms $R \to \mathcal{O}_T(T)$, and the same for $S^{-1}R$. So given two morphisms $\alpha, \beta : T \to \operatorname{Spec} S^{-1}R$ which compose with $j$ to give the same maps to $R$, then this means we have two maps $\tilde{\alpha}, \tilde{\beta} : S^{-1}R \to \mathcal{O}_T(T)$ and $\tilde{j} : R \to S^{-1}R$, then it follows from the universal property of $S^{-1}R$ that $\tilde{\alpha} = \tilde{\beta}$.

**Proposition 1.** *Monomorphisms of schemes are injective.*

*Proof.* Oddly enough, the reason for this is that the map from a fiber product to the topological fiber product is surjective. Let $j : Y \to X$ be a monomorphism. Think of $Y$ as some sort of subscheme of $X$. Consider $y, y' \in Y$ such that $j(y) = j(y') = x$. The topological fiber product $Y \times_X Y \twoheadrightarrow Y \times_X^{\text{top}} Y$, so now consider $(y, y') \in Y \times_X^{\text{top}} Y$. We have the usual diagram:




so $p_1 = p_2$ since $j$ is a monomorphism, so $Z = Y$, so $y = y'$. □

If we have a map $f : Y \to X$ and we form any kind of fiber product

$$\begin{array}{ccc} Y \times_X T & \longrightarrow & Y \\ \downarrow f' & & \downarrow f \\ T & \longrightarrow & X \end{array}$$

$f'$ is the base extension of $f$. Then it is a general category theoretic fact that $f$ being a monomorphism implies that $f'$ is a monomorphism. Therefore monomorphisms are universally injective.

Consider a map $f : Y \to X$, and $x \in X$. Then there is a fiber $f^{-1}(x)$ which is a subset of $Y$, which isn't necessarily closed since $x$ might not be closed. Now we can ask if this space is somehow a scheme.