LECTURE 32
MATH 256A

LECTURES BY: PROFESSOR MARK HAIMAN
NOTES BY: JACKSON VAN DYKE

1. Preimages

Consider a morphism $f : Y \to X$ and a point $x \in X$. Then we can consider the preimage $f^{-1}(x)$. This might not be closed, but it is certainty a subspace. Now we can ask if there is any natural way in which we can think of this as the underlying space of a scheme.

We want to think about this in terms of the map $\operatorname{Spec} k(x) \to X$ which maps $\mathrm{pt} \mapsto x$. Then we can consider the following fiber product:

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k(x) & \longrightarrow & X \end{array}$$

The topological fiber is the preimage, but the fiber as schemes might not be isomorphic to the preimage, however we do have that $Z \twoheadrightarrow P \times_X^{\text{top}} Y = f^{-1}(x)$.

In X we can take an open affine neighborhood of x, $\operatorname{Spec} R \hookrightarrow X$. Then we can further factor this map as:

$$\operatorname{Spec} k(x) \xhookrightarrow{\text{cl}} \operatorname{Spec}(\mathcal{O}_{X,x}) \xrightarrow{\text{loc}} \operatorname{Spec} R \xhookrightarrow{\text{open}} X$$

These are all monomorphisms, and compositions of monomorphisms are monomorphisms, so it follows that $Z \to Y$ is a monomorphism, which is therefore injective. Therefore Z is isomorphic to the inverse image of x. But this might not be a homeomorphism.

To see that it is, we first observe that passing to an open neighborhood of x doesn't change the problem. So we might as well assume $X = \operatorname{Spec} R$. We can also observe that Z having the subspace topology in Y is a local property, i.e. we can just check it on open elements of some affine cover of Y. Then we can the diagram:

$$\begin{array}{ccc} Z \cap W_\alpha & \longrightarrow & W_\alpha \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & X \end{array}$$

This tells us we can assume Y is affine as well.

Date: November 5, 2018.

Therefore the picture now factors as:

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \longrightarrow & \operatorname{Spec} B = Y \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k(x) & \longrightarrow & \operatorname{Spec} R_x & \longrightarrow & \operatorname{Spec} R = X \end{array}$$

But this $Z' = \operatorname{Spec}(R_x \otimes_R B) = \operatorname{Spec} S^{-1}B$. Now we have to check that the map $Z' \to \operatorname{Spec} B$ is a homeomorphism on its image. We show that every open subset $W \subseteq Z'$ is just $Z' \cap W'$ for $W' \subseteq Y$ open. But it is enough to do it for a base of open subsets, so $W = Z'_g$ suffices. Consider some $g = b/s \in S^{-1}B$. Then we have $V(g) = V(b)$, so in fact $Z'_g = Z'_b$, but we know that

$$Z'_b = j^{-1}(Y_b) = Z' \cap Y_b$$

so we are done.

1.1. **Examples.**

Example 1. Consider $\mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$ and project this to $\mathbb{A}^1_k$. For $x \in \mathbb{A}^1_k$ closed, then the preimage $f^{-1}x$ is just a homeomorphic copy of $\mathbb{A}^1_k$.

If we look at the preimage of the generic point q (which is really the prime ideal $(0) \subseteq k[t]$). The fiber over this is

$$f^{-1}(q) \cong \left(\operatorname{Spec} k(t) \otimes_{k[t]} k[t,u]\right) = \operatorname{Spec} k(t)[u] = \mathbb{A}^1_{k(t)}$$

This consists of the generic point of the whole plane, as well as the generic points of the irreducible closed subvarieties which aren't the "vertical line."

2. Relationship between classical varieties and schemes

Theorem 1. *Reduced schemes over $k = \overline{k}$ locally of finite type are*[1] *classical varieties over k.*

What kind of topological space is a general scheme? They're far from Hausdorff, but they are what is called *sober*. This means every irreducible closed subset is the closure of a unique point. Note this implies T_0. Showing this basically reduces to the case of affine schemes. In this case, the closed subsets are of the form $V(I)$ where $I \in \operatorname{Spec} R$, which we might as well assume is radical. The irreducibles are $V(P) = \overline{\{P\}}$.

On the other hand, classical varieties Y have all points closed, i.e. they are T_1, which of course implies T_0. If a space is T_1 and sober this implies the only irreducible closed subsets are singletons, so even though we aren't used to thinking of these as different things, within the world of T_0 spaces these are somehow orthogonal notions. As it turns out, we can take any space and sober it up. We can take an arbitrary space X and send it to

$$\operatorname{Sob}(X) = \{Z \subseteq X \text{ irreducible and closed}\}$$

which we give the topology such that the closed subsets are of the form

$$V(Y) = \{Z \in \operatorname{Sob}(X) \mid Z \subseteq Y\}$$

Claim 1. This is an honest topology.

[1] This is really an equivalence of categories.

Remark 1. We should probably define this officially:

Definition 1. A space Z is irreducible iff $Z \neq \emptyset$ and we don't have $Z = Z_1 \cup Z_2$ where the Z_i are proper closed subsets.

It is clear that $V(\emptyset) = \emptyset$, Even more obvious, $V(X) = \operatorname{Sob} X$ and it is also obvious that

$$V\left(\bigcap Y_\alpha\right) = \bigcap_\alpha V(Y_\alpha)$$

it also preserves finite unions $V(Y \cup Y') = V(Y) \cup V(Y')$.

It also comes with a map $i : X \to \operatorname{Sob} X$ which sends $x \mapsto \overline{\{x\}}$. Which is always irreducible. We will see next time that there is a sense in which this construction does not change the topology of X.